

Derived equivalences from cohomological approximations, and mutations of Φ -Yoneda algebras

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Abstract

In this article, a new construction of derived equivalences is given. It relates different endomorphism rings and more generally cohomological endomorphism rings - including higher extensions - of objects in triangulated categories. These objects need to be connected by certain universal maps that are cohomological approximations and that exist in very general circumstances. The construction turns out to be applicable in a wide variety of situations, covering finite dimensional algebras as well as certain infinite dimensional algebras, Frobenius categories and n -Calabi-Yau categories.

1 Introduction

Derived equivalences have become increasingly important in representation theory, Lie theory and geometry. Examples are ranging from mirror symmetry over non-commutative geometry to the Kazhdan-Lusztig conjecture and to Broué's conjecture for blocks of finite groups. In all of these situations, and in many others, derived equivalences are used that involve finite or infinite dimensional algebras. Derived equivalences between algebras, or rings, exist if and only if there exist suitable tilting complexes, as explained quite satisfactorily by Rickard's Morita theory for derived categories of rings (see [19]). Derived equivalences have been shown to preserve many significant algebraic and geometric invariants and often to provide unexpected and useful new connections.

A crucial question in this context has, however, not yet received enough answers:

How to construct derived equivalences between rings in a general setup?

A good answer - certainly not unique - to this question should be general, flexible and systematic and apply to a multitude of algebraic and geometric situations.

One well-developed approach is based on the theory of tilting modules, building upon results by Happel [8]. Other answers use ring theoretic constructions, such as trivial extensions [20].

The aim of this article is to provide a rather different approach. The input of the technology developed here is a triple of objects (X, M, Y) in a triangulated category. These objects are required to be related by certain universal maps (cohomological approximations - a new concept introduced here, continuing approximation theory of Auslander, Reiten and Smalø [1]) and some cohomological orthogonality conditions in degrees different from zero only. The output is a derived equivalence between cohomological endomorphism rings of $X \oplus M$ and of $M \oplus Y$.

The flexibility of the construction lies in the following features: We enhance endomorphism rings by higher extensions to produce cohomological endomorphism rings, broadening the classical concept of Yoneda extension algebras. Here, we can choose a set of cohomological degrees to define the cohomological endomorphism ring. Choosing degree zero only gives endomorphism rings in the usual sense - and then no orthogonality assumption is needed. Choosing all integers, or a suitable subset thereof (satisfying an associativity constraint), amplifies the concept of Yoneda extension algebras $\bigoplus_j \text{Ext}^j(S, S)$. There is also some flexibility in the choice of M .

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A special case of such a triple is given by any Auslander-Reiten triangle $X \rightarrow M \rightarrow Y$ in a derived module category; this already indicates generality of the construction. Our assumptions are actually much more general and not limited to objects in derived categories of algebras.

A particular feature of the derived equivalences constructed by this method is that they also provide a very general *mutation procedure*, turning one ring into another one in a systematic way. Tilting theory has arisen as a far reaching extension of reflection functors for quivers. Under some assumptions, but not in general, it provides mutation procedures between two given quivers or algebras, both of which are endomorphism rings of tilting modules; in the case of quivers one may reflect at sink or source vertices. Mutations similar in style also have come up in various geometric situations. The theory of cluster categories, or more generally of Calabi-Yau categories, has extended reflections to a mutation procedure, which works for representations of quivers at all vertices. Such mutations fit into the present framework as well. There is, though, a new feature introduced by our approach: Reflection does not work in general in derived categories (of quivers or algebras). Therefore cluster theory passes to the cluster category, a 'quotient' of a derived category modulo the action of some functor; endomorphism rings are taken there. In contrast to this, the current approach always produces equivalences on the level of derived categories, not just of quotient categories; throughout we are considering derived equivalences between (cohomological) endomorphism rings or quotients thereof. In the case of quivers, this possibility of passing to quotient algebras allows mutation at an arbitrary vertex.

More generality and flexibility is added by extending the concept of 'higher extensions', that is of shifted morphisms; it is possible to replace the shift functor by any other auto-equivalence of the ambient triangulated category. There is even a version using two such functors.

The main result of this article provides a construction of derived equivalences in a setup that is very general in several respects. In the following explanation we start with a special case and then add generality step by step, finally arriving at the main result.

The setup always is a triangulated category \mathcal{T} , which is an R -category for some commutative artinian ring R , with identity; so, morphism sets in \mathcal{T} are R -modules.

1. To start with, we choose any object M in \mathcal{T} and a triangle $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow X[1]$, where α and β are $\text{add}(M)$ -approximations, that is universal maps from X to objects in $\text{add}(M)$ or from $\text{add}(M)$ to Y , respectively; in particular, M_1 is in $\text{add}(M)$. For instance, Auslander-Reiten triangles (over algebras) provide such situations. If the triangle is induced by an exact sequence in an abelian category, then the theorem implies a derived equivalence between the two endomorphism rings $\text{End}_{\mathcal{T}}(X \oplus M_1)$ and $\text{End}_{\mathcal{T}}(M_1 \oplus Y)$. This can be seen as a mutation procedure relating the two endomorphism rings. The derived equivalence has already been established in [10].
2. In the second step, recasting an idea of [11], endomorphism rings are replaced by *cohomological endomorphism rings* in the following sense: Higher extensions between modules S and T are shifted morphisms in the derived category, $\text{Ext}^j(S, T) \simeq \text{Hom}(S, T[j])$. Using Yoneda multiplication of extensions, this defines an algebra structure on the cohomological endomorphism ring, or generalised Yoneda algebra, $\bigoplus_{j \in \mathbb{Z}} \text{Hom}(S, S[j])$. When S is a complex, or any object in a triangulated category \mathcal{T} , negative degrees j may occur. The main theorem provides derived equivalences between such generalised Yoneda algebras. The construction works, however, not only for these Yoneda algebras, but also for 'perforated' ones in the following sense: Choose a subset $\Phi \subset \mathbb{Z}$. Then, under some associativity constraint requiring Φ to be 'admissible' (see Subsection 2.3), the space $\bigoplus_{j \in \Phi} \text{Hom}(S, S[j])$ is an associative algebra, that in general is neither a subalgebra nor a quotient algebra of the Yoneda algebra $\bigoplus_{j \in \mathbb{Z}} \text{Hom}(S, S[j])$. This algebra is called a Φ -Yoneda algebra or a Φ -perforated Yoneda algebra. We will use the notation $E_{\mathcal{T}}^{\Phi}(Z)$ for the algebra $\bigoplus_{j \in \Phi} \text{Hom}(Z, Z[j])$, where Z is any object in \mathcal{T} .

The assumptions of the first step get modified by using cohomological approximations, in the degrees specified by Φ , instead of approximations in degree zero only. Auslander-Reiten triangles still satisfy these properties. Adding higher extensions requires also to add an orthogonality assumption without which the result would be wrong: Assume $\text{Hom}(M, X[j]) = 0 = \text{Hom}(Y, M[j])$ for all $j \in \Phi, j \neq 0$. For the sake of exposition also assume for a moment that the above triangle $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow X[1]$ is in a derived module category and it is induced from an exact sequence with corresponding properties. Then there are derived equivalences between Φ -Yoneda algebras $\mathcal{D}^b(\text{E}_{\mathcal{T}}^{\Phi}(X \oplus M)) \simeq \mathcal{D}^b(\text{E}_{\mathcal{T}}^{\Phi}(M \oplus Y))$.

3. This result needs to be modified, if the triangle is not induced by an exact sequence any more. Then some annihilators have to be factored out of the degree zero parts of the cohomological endomorphism rings, and the derived equivalences are connecting the quotient algebras $\text{E}_{\mathcal{T}}^{\Phi}(X \oplus M)/I$ and $\text{E}_{\mathcal{T}}^{\Phi}(M \oplus Y)/J$. Here, the ideals I and J can be described as follows: Let $\Gamma_0 = \text{End}_{\mathcal{T}}(M \oplus Y)$ and e the idempotent element in Γ_0 corresponding to the direct summand M . Then J is the submodule of the left Γ_0 -module $\Gamma_0 e \Gamma_0$, which is maximal with respect to $eJ = 0$. Let $\Lambda_0 = \text{End}_{\mathcal{T}}(X \oplus M)$, and f the idempotent in Λ_0 corresponding to the direct summand M . Then I is the submodule of the right Λ_0 -module $\Lambda_0 f \Lambda_0$, which is maximal with respect to $If = 0$.

Another, equivalent, description of I and J is that I consists of all elements $(x_i)_{i \in \Phi} \in \text{E}_{\mathcal{T}}^{\Phi}(X \oplus M)$ such that $x_i = 0$ for $0 \neq i \in \Phi$ and x_0 factorises through $\text{add}(M)$ and $x_0 \tilde{\alpha} = 0$, and J consists of all elements $(y_i)_{i \in \Phi} \in \text{E}_{\mathcal{T}}^{\Phi}(M \oplus Y)$ such that $y_i = 0$ for $0 \neq i \in \Phi$ and y_0 factorises through $\text{add}(M)$ and $\tilde{\beta} y_0 = 0$, where $\tilde{\alpha}$ is the diagonal morphism $\text{diag}(\alpha, 1) : X \oplus M \rightarrow M_1 \oplus M$, and $\tilde{\beta}$ is the skew-diagonal morphism $\text{skewdiag}(1, \beta) : M_1 \oplus M \rightarrow M \oplus Y$.

4. The fourth level of generalisation allows to replace the shift functor by any auto-equivalence of the triangulated category \mathcal{T} , thus providing a new and versatile meaning of 'higher extensions' in terms of morphisms with one variable shifted by powers of the auto-equivalence. The additional datum F gets mentioned, when necessary, in the notation as an additional superscript, as in $\text{E}_{\mathcal{T}}^{F, \Phi}(Z)$.

In this general form, the main theorem is as follows:

Theorem 1.1. *Let Φ be an admissible subset of \mathbb{Z} , and let \mathcal{T} be a triangulated R -category and M an object in \mathcal{T} . Assume that F is an invertible triangle functor from \mathcal{T} to itself. Suppose that*

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$$

is a triangle in \mathcal{T} such that

(1) The morphism α is a left $(\text{add}(M), F, \Phi)$ -approximation of X and β is a right $(\text{add}(M), F, -\Phi)$ -approximation of Y ,

(2) $\text{Hom}_{\mathcal{T}}(M, F^i X) = 0 = \text{Hom}_{\mathcal{T}}(F^{-i} Y, M)$ for all $0 \neq i \in \Phi$.

Then $\text{E}_{\mathcal{T}}^{F, \Phi}(X \oplus M)/I$ and $\text{E}_{\mathcal{T}}^{F, \Phi}(M \oplus Y)/J$ are derived equivalent, where I and J are the above ideals of the Φ -Yoneda algebras $\text{E}_{\mathcal{T}}^{F, \Phi}(X \oplus M)$ and $\text{E}_{\mathcal{T}}^{F, \Phi}(M \oplus Y)$, contained in $\text{End}_{\mathcal{T}}(X \oplus M)$ and $\text{End}_{\mathcal{T}}(M \oplus Y)$, respectively.

A fifth level of generalisation, using two functors F and G , will be discussed in the Appendix. A further generalisation of some results in this paper to n -angulated categories introduced in [6] will be considered in [4].

The second level of generality, where F is the shift functor and both I and J are zero, is already widely applicable. This case happens frequently for the derived category $\mathcal{D}^b(A)$ of an R -algebra A .

Corollary 1.2. *Let Φ be an admissible subset of \mathbb{N} , and let A be an R -algebra and M an A -module. If $0 \rightarrow X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow 0$ is an exact sequence in $A\text{-mod}$ such that α is a left $(\text{add}(M), \Phi)$ -approximation of X and β is a right $(\text{add}(M), -\Phi)$ -approximation of Y in $\mathcal{D}^b(A)$, and that $\text{Ext}_A^i(M, X) = 0 = \text{Ext}_A^i(Y, M)$ for all $0 \neq i \in \Phi$, then the Φ -Yoneda algebras $E_A^\Phi(X \oplus M)$ and $E_A^\Phi(M \oplus Y)$ are derived equivalent.*

These results partly generalise some results of [10].

The setup here, and the main result, covers, combines and extends several classical concepts:

Auslander algebras - endomorphism rings of direct sums of 'all' modules of an algebra of finite representation type - are the ingredients of the celebrated Auslander correspondence, characterising finite representation type via homological dimensions. Auslander algebras of derived equivalent algebras are, in general, not derived equivalent; positive results in this direction - for self-injective algebras of finite representation type - previously have been obtained in [11]. In the current approach new results can be obtained by appropriate choices of $X \oplus M$.

Another intensively studied class of algebras is that of Yoneda algebras, that is, algebras of self-extensions of a semisimple module, or more generally of any module. Apparently, the constructions in Corollary 1.2 and in [11] provide the first general class of derived equivalences for Yoneda algebras. Perforated Yoneda algebras first have been defined in [11], under the name Φ -Auslander-Yoneda algebras. The approach developed there has been based on the existence of particular kinds of derived equivalences for algebras, which then have been used to construct derived equivalences for perforated Yoneda algebras.

The main novelty of the present approach is the systematic use of cohomological data, such as cohomological approximations and perforated Yoneda algebras. This relates smoothly with a wide variety of concepts, such as Auslander-Reiten sequences and triangles, dominant dimension, Calabi-Yau categories and Frobenius categories.

The article is organised as follows. In Section 2, we first fix notation, and then recall definitions and basic results on derived equivalences as well as on admissible sets and perforated Yoneda algebras. Also, we extend the notion of \mathcal{D} -approximation to what we call cohomological \mathcal{D} -approximation with respect to (F, Φ) , where F is a functor and Φ is a subset of \mathbb{N} . In Section 3, the main result, Theorem 1.1, is proven and various easier to access situations are described, for which the assumptions of Theorem 1.1 are satisfied. Section 4 explains how Theorem 1.1 applies to a variety of situations: derived categories of Artin algebras, Frobenius categories and Calabi-Yau categories. Also, the connection to the concept of dominant dimension is explained. In Section 5, two examples are given to illustrate the results and to show the necessity of some assumptions in Theorem 1.1. In the Appendix, a more general formulation of Theorem 1.1 is stated, which involves two functors, in order to add more flexibility with a view to potential future applications.

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2 Preliminaries

In this section, we shall recall basic definitions and facts which will be needed in the proofs later on.

2.1 Conventions

Throughout this paper, R is a fixed commutative artinian ring with identity. Given an R -algebra A , by an A -module we mean a unitary left A -module; the category of all (respectively, finitely generated) A -modules is denoted by $A\text{-Mod}$ (respectively, $A\text{-mod}$), the full subcategory of $A\text{-Mod}$ consisting of all (respectively, finitely generated) projective modules is denoted by $A\text{-Proj}$ (respectively, $A\text{-proj}$). There is a similar notation for right A -modules. The stable module category $A\text{-}\underline{\text{mod}}$ of A is, by definition, the quotient category of $A\text{-mod}$ modulo the ideal generated by homomorphisms factorising through projective modules in $A\text{-proj}$. An equivalence between the stable module categories of two algebras is called a *stable equivalence*.

An R -algebra A is called an *Artin R -algebra* if A is finitely generated as an R -module. For an Artin R -algebra A , we denote by D the usual duality on $A\text{-mod}$, and by v_A the Nakayama functor $D\text{Hom}_A(-, {}_AA) : A\text{-proj} \rightarrow A\text{-inj}$. For an A -module M , we denote the first syzygy of M by $\Omega_A(M)$, and call Ω_A the *Heller loop operator* of A . The transpose of M , which is an A^{op} -module, is denoted by $\text{Tr}(M)$.

Let \mathcal{C} be an additive R -category, that is, \mathcal{C} is an additive category in which the set of morphisms between two objects in \mathcal{C} is an R -module, and the composition of morphisms in \mathcal{C} is R -bilinear. For an object X in \mathcal{C} , we denote by $\text{add}(X)$ the full subcategory of \mathcal{C} consisting of all direct summands of finite direct sums of copies of X . An object X in \mathcal{C} is called an *additive generator* for \mathcal{C} if $\mathcal{C} = \text{add}(X)$. For two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} , we write fg for their composition. For two functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ however, we write GF for the composition instead of FG .

If $f : X \rightarrow Y$ is a map between two sets X and Y , we denote the image of f by $\text{Im}(f)$. Moreover, if f is a homomorphism between two abelian groups, we denote the kernel and cokernel of f by $\text{Ker}(f)$ and $\text{Coker}(f)$, respectively.

Recall that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is invertible if there is a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ such that $GF = \text{id}_{\mathcal{C}}$ and $FG = \text{id}_{\mathcal{D}}$. In this case we write F^{-1} for G . If $\mathcal{C} = \mathcal{D}$ and if F is invertible, we say that F is an auto-equivalence. If F is a functor from \mathcal{C} to \mathcal{C} , then we write $F^0 = \text{id}_{\mathcal{C}}$, and $F^{-i} = (F^{-1})^i$ for $i > 0$ if F^{-1} exists, and $F^{-i} = 0$ otherwise.

Let \mathcal{T} be a triangulated R -category with a shift functor [1]. For two objects X and Y in \mathcal{T} , we write sometimes $\text{Ext}_{\mathcal{T}}^i(X, Y)$ for $\text{Hom}_{\mathcal{T}}(X, Y[i])$. Let Φ be a subset of \mathbb{Z} . An object M (or a full subcategory \mathcal{M}) of \mathcal{T} is called Φ -*self-orthogonal* provided that $\text{Ext}_{\mathcal{T}}^i(M, M) = 0$ (or $\text{Ext}_{\mathcal{T}}^i(\mathcal{M}, \mathcal{M}) = 0$) for all $0 \neq i \in \Phi$, where $\text{Ext}_{\mathcal{T}}^i(\mathcal{M}, \mathcal{M}) = 0$ means that $\text{Ext}_{\mathcal{T}}^i(X, Y) = 0$ for all $X, Y \in \mathcal{M}$. In case $\Phi = \mathbb{Z}$, we say that M is *self-orthogonal*. For $\Phi = \{0, 1, \dots, n\}$, we say that M is n -self-orthogonal, which is sometimes, perhaps less suggestively, referred to as n -rigid.

Replacing the shift functor by a triangle auto-equivalence F , one may also define the notion of (F, Φ) -self-orthogonality, but we refrain from introducing this notion here.

2.2 Derived equivalences

Let \mathcal{C} be an additive R -category.

By a complex X^\bullet over \mathcal{C} we mean a sequence of morphisms d_X^i between objects X^i in \mathcal{C} : $\dots \rightarrow X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} X^{i+2} \rightarrow \dots$, such that $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$; we write $X^\bullet = (X^i, d_X^i)$. For a complex X^\bullet , the *brutal truncation* $\sigma_{<i} X^\bullet$ of X^\bullet is a quotient complex of X^\bullet such that $(\sigma_{<i} X^\bullet)^k$ is X^k for all $k < i$ and zero otherwise. Similarly, we define $\sigma_{\geq i} X^\bullet$. For a fixed $n \in \mathbb{Z}$, we denote by $X^\bullet[n]$ the complex obtained from X^\bullet by shifting degree by n , that is, $(X^\bullet[n])^0 = X^n$.

The category of all complexes over \mathcal{C} with chain maps is denoted by $\mathcal{C}(\mathcal{C})$. The homotopy category of complexes over \mathcal{C} is denoted by $\mathcal{K}(\mathcal{C})$. When \mathcal{C} is an abelian category, the derived category of complexes over \mathcal{C} is denoted by $\mathcal{D}(\mathcal{C})$. The full subcategories of $\mathcal{K}(\mathcal{C})$ and $\mathcal{D}(\mathcal{C})$ consisting of bounded complexes over \mathcal{C} are denoted by $\mathcal{K}^b(\mathcal{C})$ and $\mathcal{D}^b(\mathcal{C})$, respectively. As usual, for an algebra A ,

we simply write $\mathcal{C}(A)$ for $\mathcal{C}(A\text{-mod})$, $\mathcal{K}(A)$ for $\mathcal{K}(A\text{-mod})$ and $\mathcal{K}^b(A)$ for $\mathcal{K}^b(A\text{-mod})$. Similarly, we write $\mathcal{D}(A)$ and $\mathcal{D}^b(A)$ for $\mathcal{D}(A\text{-mod})$ and $\mathcal{D}^b(A\text{-mod})$, respectively.

For an R -algebra A , the categories $\mathcal{K}(A)$ and $\mathcal{D}(A)$ are triangulated R -categories. For basic results on triangulated categories, we refer the reader to [8] and [17].

The following result, due to Rickard (see [19, Theorem 6.4]) by a direct approach, and to Keller by working in the more general setup of differential graded algebras, is fundamental in the investigation of derived equivalences.

Theorem 2.1. [19] *Let Λ and Γ be two rings. The following conditions are equivalent:*

- (a) $\mathcal{K}^-(\Lambda\text{-Proj})$ and $\mathcal{K}^-(\Gamma\text{-Proj})$ are equivalent as triangulated categories;
- (b) $\mathcal{D}^b(\Lambda\text{-Mod})$ and $\mathcal{D}^b(\Gamma\text{-Mod})$ are equivalent as triangulated categories;
- (c) $\mathcal{K}^b(\Lambda\text{-Proj})$ and $\mathcal{K}^b(\Gamma\text{-Proj})$ are equivalent as triangulated categories;
- (d) $\mathcal{K}^b(\Lambda\text{-proj})$ and $\mathcal{K}^b(\Gamma\text{-proj})$ are equivalent as triangulated categories;
- (e) Γ is isomorphic to $\text{End}_{\mathcal{K}^b(\Lambda\text{-proj})}(T^\bullet)$, where T^\bullet is a complex in $\mathcal{K}^b(\Lambda\text{-proj})$ satisfying:
 - (1) T^\bullet is self-orthogonal, that is, $\text{Hom}_{\mathcal{K}^b(\Lambda\text{-proj})}(T^\bullet, T^\bullet[i]) = 0$ for all $i \neq 0$,
 - (2) $\text{add}(T^\bullet)$ generates $\mathcal{K}^b(\Lambda\text{-proj})$ as a triangulated category.

Two rings Λ and Γ are called *derived equivalent* if the above conditions (a)-(e) are satisfied. A complex T^\bullet in $\mathcal{K}^b(\Lambda\text{-proj})$ as above is called a *tilting complex* over Λ .

For Artin algebras, the above equivalent conditions can be reformulated in terms of finitely generated modules: Two Artin R -algebras A and B are said to be *derived equivalent* if their derived categories $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories. In this case, there is a tilting complex T^\bullet in $\mathcal{K}^b(A\text{-proj})$ such that $B \simeq \text{End}_{\mathcal{K}^b(A)}(T^\bullet)$.

2.3 Admissible subsets and Φ -Yoneda algebras

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers, and let \mathbb{Z} be the set of all integers. For a natural number n or infinity, let $\mathbb{N}_n := \{i \in \mathbb{N} \mid 0 \leq i < n + 1\}$.

Recall from [11] that a subset Φ of \mathbb{Z} containing 0 is called an *admissible subset* of \mathbb{Z} if the following condition is satisfied:

If i, j and k are in Φ such that $i + j + k \in \Phi$, then $i + j \in \Phi$ if and only if $j + k \in \Phi$.

Any subset $\{0, i, j\}$ of \mathbb{N} is an admissible subset of \mathbb{Z} . Moreover, for any subset Φ of \mathbb{N} containing zero and for any positive integer $m \geq 3$, the set $\{x^m \mid x \in \Phi\}$ is admissible in \mathbb{Z} (for more examples, see [11]). Nevertheless, not every subset of \mathbb{N} containing zero is admissible, for instance, $\{0, 1, 2, 4\}$ is not admissible. In fact, this is the 'smallest' non-admissible subset of \mathbb{N} .

Admissible sets were used to define Φ -Yoneda algebras in [11], under the name of ' Φ -Auslander-Yoneda algebras'. The formulation there works more generally for monoid graded algebras. For our purpose in this paper, we restrict to the case of an object in a triangulated category.

Let Φ be an admissible subset of \mathbb{Z} , and let \mathcal{T} be a triangulated R -category with a shift functor [1]. Suppose that F is a triangle functor from \mathcal{T} to \mathcal{T} . Recall that we put $F^i = 0$ for $i < 0$ if F^{-1} does not exist.

Let $E_{\mathcal{T}}^{F, \Phi}(-, -)$ be the bi-functor

$$\begin{aligned} \bigoplus_{i \in \Phi} \text{Hom}_{\mathcal{T}}(-, F^i -) : \mathcal{T} \times \mathcal{T} &\longrightarrow R\text{-Mod}, \\ (X, Y) &\mapsto E_{\mathcal{T}}^{F, \Phi}(X, Y) := \bigoplus_{i \in \Phi} \text{Hom}_{\mathcal{T}}(X, F^i Y), \\ X \xrightarrow{f} X' &\mapsto \bigoplus_{i \in \Phi} \text{Hom}_{\mathcal{T}}(f, F^i Y), \quad Y \xrightarrow{g} Y' \mapsto \bigoplus_{i \in \Phi} \text{Hom}_{\mathcal{T}}(X, F^i g). \end{aligned}$$

Suppose that X, Y and Z are objects in \mathcal{T} . Let $(f_i)_{i \in \Phi} \in E_{\mathcal{T}}^{F, \Phi}(X, Y)$ and $(g_i)_{i \in \Phi} \in E_{\mathcal{T}}^{F, \Phi}(Y, Z)$. We define a composition as follows:

$$\begin{aligned} E_{\mathcal{T}}^{F, \Phi}(X, Y) \times E_{\mathcal{T}}^{F, \Phi}(Y, Z) &\longrightarrow E_{\mathcal{T}}^{F, \Phi}(X, Z) \\ ((f_i)_{i \in \Phi}, (g_i)_{i \in \Phi}) &\mapsto \left(\sum_{\substack{u, v \in \Phi \\ u+v=i}} f_u(F^u g_v) \right)_{i \in \Phi}. \end{aligned}$$

Since Φ is admissible, this composition is associative. Thus $E_{\mathcal{T}}^{F, \Phi}(X, X)$ is an R -algebra. It is called the Φ -Yoneda algebra or, when Φ is fixed, the *perforated Yoneda algebra* of X with respect to F . Then $E_{\mathcal{T}}^{F, \Phi}(X, Y)$ is a left $E_{\mathcal{T}}^{F, \Phi}(X, X)$ -module. When $\Phi = \mathbb{N}$, the algebra $E_{\mathcal{T}}^{F, \Phi}(X, X)$ is the orbit algebra of X under F (see [2]).

For convenience we write $E_{\mathcal{T}}^{F, \Phi}(X)$ for $E_{\mathcal{T}}^{F, \Phi}(X, X)$. In case $\mathcal{T} = \mathcal{D}^b(A)$ where A is a ring with identity, we write $E_A^{F, \Phi}(X, Y)$ for $E_{\mathcal{D}^b(A)}^{F, \Phi}(X, Y)$, and $E_A^{F, \Phi}(X)$ for $E_{\mathcal{D}^b(A)}^{F, \Phi}(X)$.

When F coincides with the shift functor, we omit the upper index F , and call $E_{\mathcal{T}}^{\Phi}(X)$ the Φ -Yoneda algebra of X , without referring to the shift functor. This is the algebra introduced in [11] and there called an Auslander-Yoneda algebra.

The following lemma is essentially taken from [11, Lemma 3.5], where a variation of it appears. The proof given there carries over to the present situation.

Lemma 2.2. *Let \mathcal{T} be a triangulated R -category with a triangle endo-functor F , and let U be an object in \mathcal{T} . Suppose that U_1, U_2 and U_3 are in $\text{add}(U)$, and that Φ is an admissible subset of \mathbb{Z} . Then,*

(1) *there is a natural isomorphism*

$$\mu : E_{\mathcal{T}}^{F, \Phi}(U_1, U_2) \longrightarrow \text{Hom}_{E_{\mathcal{T}}^{F, \Phi}(U)}(E_{\mathcal{T}}^{F, \Phi}(U, U_1), E_{\mathcal{T}}^{F, \Phi}(U, U_2)),$$

which sends $x \in E_{\mathcal{T}}^{F, \Phi}(U_1, U_2)$ to the morphism $a \mapsto ax$ for $a \in E_{\mathcal{T}}^{F, \Phi}(U, U_1)$. Moreover, if $x \in E_{\mathcal{T}}^{F, \Phi}(U_1, U_2)$ and $y \in E_{\mathcal{T}}^{F, \Phi}(U_2, U_3)$, then $\mu(xy) = \mu(x)\mu(y)$.

(2) *The functor $E_{\mathcal{T}}^{F, \Phi}(U, -) : \text{add}(U) \longrightarrow E_{\mathcal{T}}^{F, \Phi}(U)\text{-proj}$ is faithful.*

(3) *If $\text{Hom}_{\mathcal{T}}(U_1, F^i U_2) = 0$ for all $i \in \Phi \setminus \{0\}$, then the functor $E_{\mathcal{T}}^{F, \Phi}(U, -)$ induces an isomorphism of R -modules:*

$$E_{\mathcal{T}}^{F, \Phi}(U, -) : \text{Hom}_{\mathcal{T}}(U_1, U_2) \longrightarrow \text{Hom}_{E_{\mathcal{T}}^{F, \Phi}(U)}(E_{\mathcal{T}}^{F, \Phi}(U, U_1), E_{\mathcal{T}}^{F, \Phi}(U, U_2)).$$

The properties described in Lemma 2.2 will be frequently used in the proofs below.

The class of Φ -Yoneda algebras with respect to a functor includes a large class of algebras, for example the following:

(a) The endomorphism algebra of a module, in particular, the Auslander algebras of representation-finite algebras. Here we choose $\Phi = \{0\}$.

(b) The generalised Yoneda algebra of a module if we take $\Phi = \mathbb{N}$. This includes the preprojective algebras (see [2]) and the Hochschild cohomology rings of given algebras. Choosing $\Phi = 2\mathbb{N}$, we get for instance the even Hochschild cohomology rings of algebras.

(c) Certain trivial extensions: For an Artin algebra A and an A -module M we choose $\Phi = \{0, i\}$ for $i \geq 1$ an arbitrary natural number. Then $E_A^{\Phi}(M)$ is the trivial extension of $\text{End}_A(M)$ by the bimodule $\text{Ext}_A^i(M, M)$. Such rings appear naturally in the (bounded) derived category $\mathcal{D}^b(\mathbb{X})$ of coherent sheaves of a smooth projective variety \mathbb{X} over \mathbb{C} . Indeed, if X is a d -spherical object in $\mathcal{D}^b(\mathbb{X})$, then its cohomological ring $\text{End}_{\mathcal{D}^b(\mathbb{X})}^{\bullet}(X)$ is $E_{\mathcal{D}^b(\mathbb{X})}^{\{0, d\}}(X)$, this is a graded ring isomorphic to $\mathbb{C}[t]/(t^2)$ with t of degree d . For further information on spherical objects, we refer the reader to [21, Section 3c].

In general, if $\Phi = \{0, a_1, \dots, a_n\} \subseteq \mathbb{N}$ such that $a_i > 2a_{i-1}$ for $i = 2, \dots, n$, then $E_A^\Phi(X)$ is the trivial extension of $\text{End}_A(X)$ by the bimodule $\bigoplus_{0 \neq i \in \Phi} \text{Ext}_A^i(X, X)$. Note that $\Phi = \{0\} \cup \{2n+1 \mid n \in \mathbb{N}\}$ is admissible. In this case, we also get a trivial extension.

(d) The polynomial ring $R[t]$: If we take $\Phi = m\mathbb{N}$ for $m \geq 1$, then the perforated Yoneda algebra $E_{R[x]/(x^2)}^\Phi(R)$ is isomorphic to $R[t^m]$ with t a variable. If $\Phi = \{0, 1, \dots, n\}$, then $E_{R[x]/(x^2)}^\Phi(R) \simeq R[t]/(t^n)$.

2.4 \mathcal{D} -split sequences and cohomological \mathcal{D} -approximations

\mathcal{D} -split sequences have been defined in [10] in the context of constructing derived equivalences between certain endomorphism algebras. Let us recall the definition and a result in [10].

Let \mathcal{C} be an additive category and \mathcal{D} a full subcategory of \mathcal{C} . A sequence

$$X \xrightarrow{f} M \xrightarrow{g} Y$$

in \mathcal{C} is called an \mathcal{D} -split sequence if

- (1) $M \in \mathcal{D}$,
- (2) f is a left \mathcal{D} -approximation of X , and g is a right \mathcal{D} -approximation of Y , and
- (3) f is a kernel of g , and g is a cokernel of f .

Typical examples of \mathcal{D} -split sequences are Auslander-Reiten sequences. Every \mathcal{D} -split sequence provides a derived equivalence (see [10, Theorem 1.1]). Here are some details, for later reference.

Theorem 2.3. [10] *Let \mathcal{C} be an additive category, and M an object in \mathcal{C} . Suppose that*

$$X \longrightarrow M' \longrightarrow Y$$

is an $\text{add}(M)$ -split sequence in \mathcal{C} . Then the endomorphism ring $\text{End}_{\mathcal{C}}(M \oplus X)$ of $M \oplus X$ is derived-equivalent to the endomorphism ring $\text{End}_{\mathcal{C}}(M \oplus Y)$ of $M \oplus Y$ via a tilting module of projective dimension at most 1.

Now, the question arises whether Theorem 2.3 can be extended to Φ -Yoneda algebras. The second example in the final section demonstrates that this is no longer true if we just replace the endomorphism algebras in Theorem 2.3 by Φ -Yoneda algebras. Nevertheless, we shall show that under certain orthogonality conditions, there still is a positive answer. This will be discussed in detail in the next section.

The condition (3) of a \mathcal{D} -split sequence are substitutes in this general setup for requiring the short exact sequence to be exact. Since triangles in triangulated categories are replacements of short exact sequences, we may reformulate the notion of \mathcal{D} -split sequences in the following sense for triangulated categories.

Let \mathcal{T} be a triangulated category with a shift functor [1], and let \mathcal{D} be a full additive subcategory of \mathcal{T} . A triangle

$$X \xrightarrow{\alpha} M' \xrightarrow{\beta} Y \longrightarrow X[1]$$

in \mathcal{T} is called a \mathcal{D} -split triangle if $M' \in \mathcal{D}$, the map α is a left \mathcal{D} -approximation of X and the map β is a right \mathcal{D} -approximation of Y .

Thus, for an Artin R -algebra A , every \mathcal{D} -split sequence in $A\text{-mod}$ extends to a \mathcal{D} -split triangle in $\mathcal{D}^b(A)$.

Next, we introduce the left and right cohomological \mathcal{D} -approximations with respect to (F, Φ) , which generalise the notions of left and right \mathcal{D} -approximations, respectively.

Suppose that \mathcal{C} is a category with an endo-functor $F : \mathcal{C} \rightarrow \mathcal{C}$. Let \mathcal{D} be a full subcategory of \mathcal{C} , and let Φ be a non-empty subset of \mathbb{N} . If F has an inverse, then Φ may be chosen to be a subset of \mathbb{Z} . Suppose that X is an object of \mathcal{C} . A morphism $f : X \rightarrow D$ in \mathcal{C} is called a *left cohomological \mathcal{D} -approximation* of X with respect to (F, Φ) (or shortly, a left (\mathcal{D}, F, Φ) -approximation of X) if $D \in \mathcal{D}$, and for any morphism $g : X \rightarrow F^i(D')$ with $D' \in \mathcal{D}$ and $i \in \Phi$, there is a morphism $g' : D \rightarrow F^i(D')$ such that $g = fg'$. Here $F^0 = \text{id}_{\mathcal{C}}$. Similarly, we have the notion of a right (\mathcal{D}, F, Φ) -approximation of X in \mathcal{T} , that is, a morphism $f : D \rightarrow X$ with D in \mathcal{D} is called a right (\mathcal{D}, F, Φ) -approximation of X if, for any $i \in \Phi$ and any morphism $g : F^i D' \rightarrow X$ with D' in \mathcal{D} , there is a morphism $g' : F^i D' \rightarrow D$ such that $g = g'f$.

Note that if $F = \text{id}_{\mathcal{C}}$ and $\Phi = \{0\}$, then we get the original notion of approximations in the sense of Auslander and Smalø. (In ring theory, such approximations are called pre-envelope and precover, respectively). Moreover, if $0 \in \Phi$, then every left (\mathcal{D}, F, Φ) -approximation of X is also a left \mathcal{D} -approximation of X , and every right (\mathcal{D}, F, Φ) -approximation of X is also a right \mathcal{D} -approximation of X .

If $F = [1]$ and $\mathcal{T} = \mathcal{D}^b(A)$ for an Artin algebra A , then $\text{Hom}_{\mathcal{T}}(X, F^i Y) \simeq \text{Ext}_A^i(X, Y)$ for all $X, Y \in A\text{-mod}$ and all $i \geq 0$. For this reason, a (\mathcal{D}, F, Φ) -approximation has been called a *cohomological approximation*.

In this paper, we are mainly interested in the case where \mathcal{C} is a triangulated R -category \mathcal{T} with an endo-functor F , and \mathcal{D} is a full subcategory of \mathcal{T} . Thus, a morphism $f : X \rightarrow D$ with $D \in \mathcal{D}$ and $X \in \mathcal{T}$ is a left (\mathcal{D}, F, Φ) -approximation of X if and only if the canonical map $\text{E}_{\mathcal{T}}^{F, \Phi}(f, D') : \text{E}_{\mathcal{T}}^{F, \Phi}(D, D') \rightarrow \text{E}_{\mathcal{T}}^{F, \Phi}(X, D')$, defined by $(x_i)_{i \in \Phi} \mapsto (fx_i)_{i \in \Phi}$, is surjective for all $D' \in \mathcal{D}$. Similarly, a morphism $g : D \rightarrow X$ with $D \in \mathcal{D}$ and $X \in \mathcal{T}$ is a right (\mathcal{D}, F, Φ) -approximation of X if and only if the canonical map $\text{Hom}_{\mathcal{T}}(F^j D', g) : \text{Hom}_{\mathcal{T}}(F^j D', D) \rightarrow \text{Hom}_{\mathcal{T}}(F^j D', X)$ is surjective for every $D' \in \mathcal{D}$ and $j \in \Phi$. If, moreover, F is an invertible triangle functor, then a morphism $g : D \rightarrow X$ with $D \in \mathcal{D}$ and $X \in \mathcal{T}$ is a right (\mathcal{D}, F, Φ) -approximation of X if and only if the canonical map $\text{E}_{\mathcal{T}}^{F, -\Phi}(D', g) : \text{E}_{\mathcal{T}}^{F, -\Phi}(D', D) \rightarrow \text{E}_{\mathcal{T}}^{F, -\Phi}(D', X)$ is surjective for all $D' \in \mathcal{D}$. Note that here we need the minus sign for Φ and that F^{-1} exists.

If F coincides with the shift functor $[1]$, we simply speak of (\mathcal{D}, Φ) -approximations, without mentioning F .

Note that if Φ contains zero and if $\text{Hom}_{\mathcal{T}}(X, F^i D') = 0$ for all $0 \neq i \in \Phi$ and $D' \in \mathcal{D}$, then f is a left (\mathcal{D}, F, Φ) -approximation of X if and only if f is a left \mathcal{D} -approximation of X . A dual statement is also true for a right (\mathcal{D}, F, Φ) -approximation of X .

Here is a source of examples of (\mathcal{D}, Φ) -approximations. Suppose that $\mathcal{T} = \mathcal{D}^b(A)$ for A an Artin R -algebra and that Φ is a subset of \mathbb{Z} . Let $X \xrightarrow{\alpha} M \xrightarrow{\beta} Y \rightarrow X[1]$ be an Auslander-Reiten triangle in \mathcal{T} . If neither X nor Y belongs to $\text{add}(M[i])$ for every $0 \neq i \in \Phi$, then α is a left $(\text{add}(M), \Phi)$ -approximation of X , and β is a right $(\text{add}(M), \Phi)$ -approximation of Y .

Finally, we note the difference of a left (\mathcal{D}, F, Φ) -approximation of X from a left $(\cup_{i \in \Phi} F^i \mathcal{D})$ -approximation of X in the sense of Auslander and Smalø, where $\cup_{i \in \Phi} F^i \mathcal{D}$ is the full subcategory of \mathcal{T} with all objects in $F^i \mathcal{D}$ for all $i \in \Phi$. Suppose $0 \in \Phi$. Then a (\mathcal{D}, F, Φ) -approximation is a $(\cup_{i \in \Phi} F^i \mathcal{D})$ -approximation, but the converse is not true in general. If $0 \notin \Phi$, then the two concepts are independent. So, roughly speaking, a cohomological \mathcal{D} -approximation with respect to (F, Φ) emphasises not only the factorisations but also that the object belongs to the given subcategory \mathcal{D} (and not to $F^i \mathcal{D}$ for $0 \neq i \in \Phi$).

3 Derived equivalences for Φ -Yoneda algebras

In this section, we shall prove Theorem 1.1 and derive some consequences and some simplifications in special cases.

Suppose that \mathcal{T} is a triangulated R -category with a shift functor $[1]$, and M is an object in \mathcal{T} . Suppose that F is an auto-equivalence of \mathcal{T} , which may be different from the shift functor.

For a subset Φ of \mathbb{Z} , we define $-\Phi := \{-x \mid x \in \Phi\}$, and

$$\mathcal{X}_{\mathcal{T}}^{F,\Phi}(M) = \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(X, F^i M) = 0 \text{ for all } i \in \Phi \setminus \{0\}\},$$

$$\mathcal{Y}_{\mathcal{T}}^{F,\Phi}(M) = \{Y \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(M, F^i Y) = 0 \text{ for all } i \in \Phi \setminus \{0\}\}.$$

Let n be a positive integer. For simplicity, we write $\mathcal{X}_{\mathcal{T}}^{F,n}(M)$ for $\mathcal{X}_{\mathcal{T}}^{F,\{0,1,2,\dots,n\}}(M)$, and $\mathcal{X}_{\mathcal{T}}^{F,\infty}(M)$ for $\mathcal{X}_{\mathcal{T}}^{F,\mathbb{N}}(M)$ if \mathcal{T} is clear in the context. Similarly, the notations $\mathcal{Y}_{\mathcal{T}}^{F,n}(M)$ and $\mathcal{Y}_{\mathcal{T}}^{F,\infty}(M)$ are defined.

As usual, F is omitted in notation when it coincides with the shift functor.

Given a triangle $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ in \mathcal{T} with $M_1 \in \text{add}(M)$, we define

$$\tilde{w} = (w, 0) : Y \longrightarrow (X \oplus M)[1], \quad \bar{w} = (0, w)^T : M \oplus Y \longrightarrow X[1],$$

where $(0, w)^T$ stands for the transpose of the matrix $(0, w)$, and

$$I := \{x = (x_i) \in E_{\mathcal{T}}^{F,\Phi}(X \oplus M) \mid x_i = 0 \text{ for } 0 \neq i \in \Phi, x_0 \text{ factorises through } \text{add}(M) \text{ and } \tilde{w}[-1]\},$$

$$J := \{y = (y_i) \in E_{\mathcal{T}}^{F,\Phi}(M \oplus Y) \mid y_i = 0 \text{ for } 0 \neq i \in \Phi, y_0 \text{ factorises through } \text{add}(M) \text{ and } \bar{w}\}.$$

The sets I and J are indeed independent of F and $\Phi \setminus \{0\}$, and contained in $\text{End}_{\mathcal{T}}(X \oplus M)$ and $\text{End}_{\mathcal{T}}(M \oplus Y)$, respectively.

The main result of this paper is the following theorem which is a reformulation of Theorem 1.1.

Theorem 3.1. *Let Φ be an admissible subset of \mathbb{Z} , let \mathcal{T} be a triangulated R -category with an auto-equivalence F , and let M be an object in \mathcal{T} . Suppose that*

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$$

is a triangle in \mathcal{T} such that the morphism α is a left $(\text{add}(M), F, \Phi)$ -approximation of X , that the morphism β is a right $(\text{add}(M), F, -\Phi)$ -approximation of Y and that $X \in \mathcal{Y}_{\mathcal{T}}^{F,\Phi}(M)$ and $Y \in \mathcal{X}_{\mathcal{T}}^{F,\Phi}(M)$. Then the algebras $E_{\mathcal{T}}^{F,\Phi}(X \oplus M)/I$ and $E_{\mathcal{T}}^{F,\Phi}(M \oplus Y)/J$ are derived equivalent.

Proof. Let $V = X \oplus M$ and $W = M \oplus Y$. Set

$$\bar{\alpha} := (\alpha, 0) : X \rightarrow M_1 \oplus M, \quad \bar{\beta} := \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} : M_1 \oplus M \rightarrow M \oplus Y, \quad \bar{w} := \begin{pmatrix} 0 \\ w \end{pmatrix} : M \oplus Y \rightarrow X[1];$$

$$\tilde{\alpha} := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} : X \oplus M \rightarrow M_1 \oplus M, \quad \tilde{\beta} := \begin{pmatrix} \beta \\ 0 \end{pmatrix} : M_1 \oplus M \rightarrow Y, \quad \tilde{w} := (w, 0) : Y \longrightarrow (X \oplus M)[1].$$

Then there are two triangles in \mathcal{T} :

$$X \xrightarrow{\bar{\alpha}} M_1 \oplus M \xrightarrow{\bar{\beta}} W \xrightarrow{\bar{w}} X[1],$$

$$Y[-1] \xrightarrow{-\tilde{w}[-1]} V \xrightarrow{\tilde{\alpha}} M_1 \oplus M \xrightarrow{\tilde{\beta}} Y.$$

Since F is a triangle functor, there is a natural isomorphism $\delta : F[1] \rightarrow [1]F$. That is, for any object X in \mathcal{T} , there is an isomorphism $\delta_X : F(X[1]) \rightarrow (FX)[1]$, which is natural in X . The isomorphism $F^i(X[j]) \rightarrow (F^i X)[j]$ is denoted by $\delta(F, i, X, j)$. In part II of this article, there will be further discussion of these natural transformations.

Lemma 3.2. (1) For any morphism $x_i : V \rightarrow F^i V$ with $i \in \Phi$, there is a morphism $t_i : Y[-1] \rightarrow (F^i Y)[-1]$ such that $(\tilde{w}[-1])x_i = t_i \delta(F, i, Y, -1)^{-1} (F^i(\tilde{w}[-1]))$.

(2) For any morphism $y_i : W \rightarrow F^i W$ with $i \in \Phi$, there is a morphism $t_i : X[1] \rightarrow (F^i X)[1]$ such that $y_i(F^i \tilde{w})\delta(F, i, X, 1) = \tilde{w}t_i$.

Proof. (1) Note that $\tilde{\alpha}$ is a left $(\text{add}(M), F, \Phi)$ -approximation of V . Thus, given $x_i : V \rightarrow F^i V$, there is a morphism $y_i : M_1 \oplus M \rightarrow F^i(M_1 \oplus M)$ such that $\tilde{\alpha}y_i = x_i(F^i \tilde{\alpha})$. Since F is a triangle functor, the second triangle implies that there is a triangle (see [8, p.4])

$$(F^i Y)[-1] \xrightarrow{\delta(F, i, Y, -1)^{-1}(-F^i(\tilde{w}[-1]))} F^i V \xrightarrow{F^i \tilde{\alpha}} F^i(M_1 \oplus M) \xrightarrow{F^i \tilde{\beta}} F^i Y.$$

Thus there is a morphism $t_i : Y[-1] \rightarrow (F^i Y)[-1]$ such that $(\tilde{w}[-1])x_i = t_i \delta(F, i, Y, -1)^{-1} (F^i(\tilde{w}[-1]))$.

(2) The proof of (2) is similar to that of (1), using the following triangle

$$F^i X \xrightarrow{F^i \tilde{\alpha}} F^i(M_1 \oplus M) \xrightarrow{F^i \tilde{\beta}} F^i W \xrightarrow{(F^i \tilde{w})\delta(F, i, X, 1)} (F^i X)[1].$$

□

Now we prove that the quotient rings in Theorem 3.1 are well-defined.

Lemma 3.3. The I and J appearing in Theorem 3.1 are ideals of $E_{\mathcal{T}}^{F, \Phi}(V)$ and $E_{\mathcal{T}}^{F, \Phi}(W)$, respectively.

Proof. By definition, a morphism $f : V \rightarrow V$ factorises through $\text{add}(M)$ if and only if there is an object M' in $\text{add}(M)$ and there are two morphisms $f_1 : V \rightarrow M'$ and $f_2 : M' \rightarrow V$ such that $f = f_1 f_2$. A morphism $g : V \rightarrow V$ factorises through $\tilde{w}[-1] : Y[-1] \rightarrow V$ if and only if there is a morphism $g' : V \rightarrow Y[-1]$ such that $g = g'(\tilde{w}[-1])$. In the following, we shall prove that I is an ideal in $E_{\mathcal{T}}^{F, \Phi}(V)$.

The set I is closed under addition in $E_{\mathcal{T}}^{F, \Phi}(V)$. To show that I is a two-sided ideal in $E_{\mathcal{T}}^{F, \Phi}(V)$, we pick an $x = (x_i)_{i \in \Phi} \in I$ and a $y = (y_i)_{i \in \Phi} \in E_{\mathcal{T}}^{F, \Phi}(V)$, and calculate the products xy and yx in $E_{\mathcal{T}}^{F, \Phi}(V)$. Note that $xy = (x_0 y_i)_{i \in \Phi}$ and $yx = (y_i F^i x_0)_{i \in \Phi}$ since $x_i = 0$ for $0 \neq i \in \Phi$. We write $x_0 = uv$ for $u : V \rightarrow M'$ and $v : M' \rightarrow V$, where M' is an object in $\text{add}(M)$, and $x_0 = s(\tilde{w}[-1])$ for a morphism $s : V \rightarrow Y[-1]$.

We first show that I is a right ideal.

(1) Let $i = 0$. The map $x_0 y_0$ factorises through an object in $\text{add}(M)$. Since x_0 factorises through $\tilde{w}[-1]$, it follows from Lemma 3.2 (1) that $x_0 y_0$ factorises also through $\tilde{w}[-1]$.

(2) Let $0 \neq i \in \Phi$. In this case, $\text{Hom}_{\mathcal{T}}(M, F^i X) = 0$ by the assumption $X \in \mathcal{Y}^{F, \Phi}(M)$. Let p_X and p_M be the projections of V onto X and M , respectively. Then the composition $vy_i F^i p_X : M' \xrightarrow{v} V \xrightarrow{y_i} F^i V \xrightarrow{F^i p_X} F^i X$ belongs to $\text{Hom}_{\mathcal{T}}(M', F^i X) = 0$. Thus $x_0 y_i F^i p_X = uv y_i F^i p_X = 0$. By Lemma 3.2 (1), there is a morphism $t_i : Y[-1] \rightarrow F^i Y[-1]$ such that $(\tilde{w}[-1])y_i = t_i \delta(F, i, Y, -1)^{-1} F^i(\tilde{w}[-1])$. Hence

$$\begin{aligned} x_0 y_i (F^i p_M) &= s(\tilde{w}[-1])y_i (F^i p_M) = s t_i \delta(F, i, Y, -1)^{-1} F^i(\tilde{w}[-1]) (F^i p_M) \\ &= s t_i \delta(F, i, Y, -1)^{-1} F^i(\tilde{w}[-1] p_M) \\ &= s t_i \delta(F, i, Y, -1)^{-1} F^i((w[-1], 0) \begin{pmatrix} 0 \\ 1_M \end{pmatrix}) = 0. \end{aligned}$$

Altogether, $x_0 y_i = x_0 y_i (F^i p_X, F^i p_M) = 0$ for $0 \neq i \in \Phi$.

Hence $xy \in I$, and I is a right ideal in $E_{\mathcal{T}}^{F, \Phi}(V)$.

Next, we show that I is a left ideal, that is, we check $(y_i F^i x_0)_{i \in \Phi} \in I$.

(3) The map $y_0 x_0$ factorises through an object in $\text{add}(M)$ and through $\tilde{w}[-1]$.

(4) Let $0 \neq i \in \Phi$. Note that $\tilde{\alpha} : V \rightarrow M_1 \oplus M$ is a left $(\text{add}(M), F, \Phi)$ -approximation of V . Thus there is a morphism $h_i : M_1 \oplus M \rightarrow F^i(M')$ such that $y_i(F^i u) = \tilde{\alpha} h_i$. By assumption, $\text{Hom}_{\mathcal{T}}(M, F^i X) = 0$. This

implies that $h_i(F^i v)(F^i p_X) = 0$, and therefore $y_i(F^i x_0)(F^i p_X) = 0$. Since $(F^i \tilde{w}[-1])(F^i p_M) = 0$, we get $y_i(F^i x_0)(F^i p_M) = 0$. Thus $y_i F^i x_0 = 0$ for $0 \neq i \in \Phi$.

Hence $yx \in I$, and I is a left ideal in $E_{\mathcal{T}}^{F,\Phi}(V, \cdot)$. Thus I is an ideal in $E_{\mathcal{T}}^{F,\Phi}(V)$.

Similarly, J is an ideal in $E_{\mathcal{T}}^{F,\Phi}(W)$. \square

We know that $E_{\mathcal{T}}^{F,\Phi}(V, Z)$ is a $E_{\mathcal{T}}^{F,\Phi}(V)$ -module for any object Z in \mathcal{T} . The next lemma shows that the ideal I of $E_{\mathcal{T}}^{F,\Phi}(V)$ may annihilate some modules of this form.

Lemma 3.4. *Keep the notations as above. Then*

- (1) $I \cdot E_{\mathcal{T}}^{F,\Phi}(V, M) = 0$.
- (2) $I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X) = \{(x_i)_{i \in \Phi} \in E_{\mathcal{T}}^{F,\Phi}(V, X) \mid x_i = 0 \text{ for } 0 \neq i \in \Phi, x_0 \text{ factorises through } \text{add}(M) \text{ and } w[-1]\}$.
- (3) *For $x = (x_i)_{i \in \Phi} \in E_{\mathcal{T}}^{F,\Phi}(V', X)$ with $V' \in \text{add}(V)$, we have $\text{Im}(\mu(x)) \subseteq I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$ if and only if $x_i = 0$ for all $0 \neq i \in \Phi$ and x_0 factorises through $\text{add}(M)$ and $w[-1]$, where μ is defined in Lemma 2.2 (1).*
- (4) *Let $f : M' \rightarrow X$ with $M' \in \text{add}(M)$. Then $\text{Im}(E_{\mathcal{T}}^{F,\Phi}(V, f)) \subseteq I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$ if and only if f factorises through $w[-1]$.*

Proof. (1) We denote by $\lambda_M = (0, 1) : M \rightarrow V$ the canonical inclusion. Let $(x_i)_{i \in \Phi} \in I$ and $(y_i)_{i \in \Phi} \in E_{\mathcal{T}}^{F,\Phi}(V, M)$. Then $(x_i)(y_i) = (x_0 y_i)_{i \in \Phi}$ since $x_i = 0$ for $0 \neq i \in \Phi$. Since I is an ideal in $E_{\mathcal{T}}^{F,\Phi}(V)$, it follows that $x(y_i(F^i \lambda_M))_{i \in \Phi} = (x_0 y_i(F^i \lambda_M))_{i \in \Phi} \in I$. By the definition of I , we have $x_0 y_i(F^i \lambda_M) = 0$ for all $0 \neq i \in \Phi$ and $x_0 y_0 \lambda_M$ factorises through $\tilde{w}[-1]$. Moreover, $x_0 y_0 \lambda_M = (x_0 y_0 \lambda_M p_M) \lambda_M = s(\tilde{w}[-1] p_M) \lambda_M = s \cdot 0 \cdot \lambda_M = 0$, where s is a morphism from V to $Y[-1]$. Hence $x_0 y_i(F^i \lambda_M) = 0$, and $x_0 y_i = x_0 y_i(F^i \lambda_M)(F^i p_M) = 0 \cdot F^i p_M = 0$ for all $i \in \Phi$. Thus (1) follows.

(2) Let $\lambda_X : X \rightarrow V$ be the canonical inclusion. As in case (1), it follows that, for $(x_i)_{i \in \Phi} \in I$ and $(y_i)_{i \in \Phi} \in E_{\mathcal{T}}^{F,\Phi}(V, X)$, we have $(x_i)(y_i) = (x_0 y_i)_{i \in \Phi}$, and that $x_0 y_0 \lambda_X$ factorises through $\tilde{w}[-1]$ and $\text{add}(M)$. Hence $x_0 y_0 = (x_0 y_0 \lambda_X) p_X = s(\tilde{w}[-1] p_X) = s(w[-1])$, where s is a morphism from V to $Y[-1]$. Conversely, let $x = (x_i)_{i \in \Phi} \in E_{\mathcal{T}}^{F,\Phi}(V, X)$, and suppose that $x_i = 0$ for all $0 \neq i \in \Phi$ and that x_0 factorises through $\text{add}(M)$ and $w[-1]$. For $f : U \rightarrow Z$ in \mathcal{T} , we denote by \underline{f} the element of $E_{\mathcal{T}}^{F,\Phi}(U, Z)$ concentrated only in degree $0 \in \Phi$. Then it is straightforward to check that $x \underline{\lambda_X}$ belongs to I . Thus, $x = x \underline{\lambda_X} p_X \in I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$.

(3) First, suppose $V' = V$ and $\text{Im}(\mu(x)) \subseteq I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$. Then x , the image of 1_V under $\mu(x)$, belongs to $I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$. Thus, by (2), we have $x_i = 0$ for all $0 \neq i \in \Phi$ and that x_0 factorises through $\text{add}(M)$ and $w[-1]$. Conversely, suppose that $x \in I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$. Then, for any $y \in E_{\mathcal{T}}^{F,\Phi}(V)$, the image of y under $\mu(x)$ is $y \cdot x$. Since $I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$ is a $E_{\mathcal{T}}^{F,\Phi}(V)$ -submodule of $E_{\mathcal{T}}^{F,\Phi}(V, X)$, we have $yx \in I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$.

Secondly, suppose that V' is a direct sum of n copies of V , and $x \in E_{\mathcal{T}}^{F,\Phi}(V', X)$. We identify $E_{\mathcal{T}}^{F,\Phi}(V', X)$ with $\bigoplus_{i=1}^n E_{\mathcal{T}}^{F,\Phi}(V, X)$, and write $x = (a_1, \dots, a_n)^T$, a column matrix with $a_i \in E_{\mathcal{T}}^{F,\Phi}(V, X)$. Then the image of $\mu(x)$ is the sum of the image of $\mu(a_i)$ for $1 \leq i \leq n$. Now the conclusion follows from the first case.

Finally, suppose that V' is a direct summand of n copies of V , that is, $\bigoplus_{i=1}^n V = V' \oplus V''$. If $x \in E_{\mathcal{T}}^{F,\Phi}(V', X)$, then we may consider $(x, 0)^T$ as an element in $E_{\mathcal{T}}^{F,\Phi}(\bigoplus_{i=1}^n V, X)$. Then the proof is reduced to the second case.

(4) follows from (3) because of $E_{\mathcal{T}}^{F,\Phi}(V, f) = \mu(\underline{f})$. \square

Let \tilde{T}^\bullet be the complex

$$\tilde{T}^\bullet : 0 \longrightarrow E_{\mathcal{T}}^{F,\Phi}(V, X) \xrightarrow{E_{\mathcal{T}}^{F,\Phi}(V, \tilde{\alpha})} E_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M) \longrightarrow 0,$$

where the term $E_{\mathcal{T}}^{F,\Phi}(V, X)$ is in degree zero. Then it is the direct sum of the following two complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_{\mathcal{T}}^{F,\Phi}(V, X) & \xrightarrow{E_{\mathcal{T}}^{F,\Phi}(V, \alpha)} & E_{\mathcal{T}}^{F,\Phi}(V, M_1) & \longrightarrow & 0, \\ 0 & \longrightarrow & 0 & \longrightarrow & E_{\mathcal{T}}^{F,\Phi}(V, M) & \longrightarrow & 0. \end{array}$$

Let $P = E_{\mathcal{T}}^{F,\Phi}(V, X)/I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$, and let $p : E_{\mathcal{T}}^{F,\Phi}(V, X) \rightarrow P$ be the canonical surjection. Then, by Lemma 3.4 (1), we may write $E_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha}) = pq$ with $q : P \rightarrow E_{\mathcal{T}}^{F,\Phi}(V, X)$. The complex

$$T^\bullet : \quad 0 \longrightarrow P \longrightarrow E_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M) \longrightarrow 0$$

in $\mathcal{D}^b(E_{\mathcal{T}}^{F,\Phi}(V)/I)$ is the direct sum of the complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{q} & E_{\mathcal{T}}^{F,\Phi}(V, M_1) & \longrightarrow & 0, \\ 0 & \longrightarrow & 0 & \longrightarrow & E_{\mathcal{T}}^{F,\Phi}(V, M) & \longrightarrow & 0. \end{array}$$

Each term of T^\bullet is a finitely generated projective $E_{\mathcal{T}}^{F,\Phi}(V)/I$ -module.

Before proceeding further, we need to introduce some more notation. Set

$$\Lambda := E_{\mathcal{T}}^{F,\Phi}(V), \quad \Gamma := E_{\mathcal{T}}^{F,\Phi}(W), \quad \bar{\Lambda} := \Lambda/I, \quad \bar{\Gamma} := \Gamma/J,$$

where I and J are defined just before Theorem 3.1.

Lemma 3.5. *T^\bullet is a tilting complex over $\bar{\Lambda}$.*

Proof. It is clear that $\text{Hom}_{\mathcal{K}^b(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[i]) = 0$ for $i \leq -2$ and for $i \geq 2$. We have to check that $\text{Hom}_{\mathcal{K}^b(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[1]) = 0$ and $\text{Hom}_{\mathcal{K}^b(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[-1]) = 0$.

Let $f^\bullet \in \text{Hom}_{\mathcal{K}^b(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[1])$. Consider the following diagram:

$$\begin{array}{ccccccc} & & E_{\mathcal{T}}^{F,\Phi}(V, X) & & & & \\ & & \downarrow p & & & & \\ 0 & \longrightarrow & P & \xrightarrow{q} & E_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M) & \longrightarrow & 0 \\ & & \downarrow f^0 & & \downarrow & & \\ 0 & \longrightarrow & P & \xrightarrow{q} & E_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M) & \longrightarrow & 0 \end{array}$$

Since both X and $M_1 \oplus M$ are in $\text{add}(V)$, Lemma 2.2 (1) provides an isomorphism $\mu : E_{\mathcal{T}}^{F,\Phi}(X, M_1 \oplus M) \simeq \text{Hom}_{\Lambda}(E_{\mathcal{T}}^{F,\Phi}(V, X), E_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M))$ and an element $u = (u_i)_{i \in \Phi} \in E_{\mathcal{T}}^{F,\Phi}(X, M_1 \oplus M)$ such that $pf^0 = \mu(u)$. By assumption, $\bar{\alpha}$ is a left $(\text{add}(M), F, \Phi)$ -approximation of X . This yields for each $i \in \Phi$ a morphism $u'_i : M_1 \oplus M \rightarrow F^i(M_1 \oplus M)$ such that $u_i = \bar{\alpha}u'_i$. Clearly, $u' := (u'_i)_{i \in \Phi} \in E_{\mathcal{T}}^{F,\Phi}(M_1 \oplus M, M_1 \oplus M)$, and $\mu(u') \in \text{Hom}_{\Lambda}(E_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M), E_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M))$. Now, we have to check the following diagram is commutative:

$$\begin{array}{ccc} E_{\mathcal{T}}^{F,\Phi}(V, X) & \xrightarrow{E_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha})} & E_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M) \\ \mu(u) \downarrow & & \downarrow \mu(u') \\ E_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M) & \xlongequal{\quad} & E_{\mathcal{T}}^{F,\Phi}(V, M_1 \oplus M) \end{array}$$

In fact, if $a = (a_j)_{j \in \Phi} \in E_T^{F, \Phi}(V, X)$, then it is sent to $b := (a_j F^j(\bar{\alpha}))_{j \in \Phi}$ by $E_T^{F, \Phi}(V, \bar{\alpha})$, and further sent to $bu' = (a_j (F^j \bar{\alpha}))_{j \in \Phi} u'$ by $\mu(u')$. An easy calculation shows that $bu' = au$, the image of a under $\mu(u)$. Thus the diagram is commutative, and

$$pf^0 = \mu(u) = E_T^{F, \Phi}(V, \bar{\alpha})\mu(u') = pq\mu(u').$$

This means that $f^0 = q\mu(u')$ - since p is surjective - and that $f^\bullet = 0$ in $\mathcal{K}^b(\bar{\Lambda}\text{-proj})$. Therefore $\text{Hom}_{\mathcal{K}^b(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[1]) = 0$.

Let $f^\bullet \in \text{Hom}_{\mathcal{K}^b(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[-1])$. Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{q} & E_T^{F, \Phi}(V, M_1 \oplus M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f^1 & & \downarrow \\ & & 0 & \longrightarrow & P & \xrightarrow{q} & E_T^{F, \Phi}(V, M_1 \oplus M) \longrightarrow 0 \end{array}$$

Since p is surjective and $E_T^{F, \Phi}(V, M_1 \oplus M)$ is projective in $\Lambda\text{-mod}$, f^1 can be lifted along p , say $f^1 = gp$ with $g : E_T^{F, \Phi}(V, M_1 \oplus M) \rightarrow E_T^{F, \Phi}(V, X)$. By assumption, we have $X \in \mathcal{B}^{F, \Phi}(M)$, and, by Lemma 2.2 (3), there is a homomorphism $u : M_1 \oplus M \rightarrow X$ such that $g = E_T^{F, \Phi}(V, u)$. Thus

$$E_T^{F, \Phi}(V, u\bar{\alpha}) = E_T^{F, \Phi}(V, u)E_T^{F, \Phi}(V, \bar{\alpha}) = gpq = f^1q = 0.$$

Lemma 2.2 (2) implies $u\bar{\alpha} = 0 = u\alpha$. Therefore u factorises through $-w[-1]$. By Lemma 3.4 (4), the image of $g (= E_T^{F, \Phi}(V, u))$ is contained in $I \cdot E_T^{F, \Phi}(V, X)$. It follows that $f^1 = gp = 0$ and $f^\bullet = 0$. Hence $\text{Hom}_{\mathcal{K}^b(\bar{\Lambda}\text{-proj})}(T^\bullet, T^\bullet[-1]) = 0$.

Now, $\text{add}(T^\bullet)$ generates $\mathcal{K}^b(\bar{\Lambda}\text{-proj})$ as a triangulated category. Thus T^\bullet is a tilting complex over $\bar{\Lambda}$. \square

Remark. To get a tilting complex from \tilde{T}^\bullet , one may consider the ideal I_0 of $E_T^\Phi(V)$ consisting of all endomorphisms $V \rightarrow V$ which are of the form fg with $f : V \rightarrow M'$ and $g : M' \rightarrow V$ such that $M' \in \text{add}(M)$ and $g\bar{\alpha} = 0$. Then it is easy to show that the quotient complex of \tilde{T}^\bullet modulo $I_0\tilde{T}^\bullet$ is a two-term tilting complex over $E_T^\Phi(V)/I_0$. We will not use this complex because its endomorphism algebra cannot be described in a nice way. Note that the ideal I_0 of $E_T^\Phi(V)$ is properly contained in I in general.

Lemma 3.6. *The two rings $\bar{\Gamma}$ and $\text{End}_{\mathcal{K}^b(\bar{\Lambda}\text{-proj})}(T^\bullet)$ are isomorphic.*

Proof. Since $\bar{\Lambda}$ is a quotient algebra of Λ , the category $\bar{\Lambda}\text{-mod}$ can be viewed as a full subcategory of $\Lambda\text{-mod}$, and it follows that $\mathcal{K}^b(\bar{\Lambda})$ can be viewed as a full subcategory of $\mathcal{K}^b(\Lambda)$. Thus, we have an isomorphism $\text{End}_{\mathcal{K}^b(\bar{\Lambda}\text{-proj})}(T^\bullet) \simeq \text{End}_{\mathcal{K}^b(\Lambda)}(T^\bullet)$. To prove the lemma, we shall construct an isomorphism from $\text{End}_{\mathcal{K}^b(\Lambda)}(T^\bullet)$ to $\bar{\Gamma}$.

Let $f^\bullet \in \text{End}_{\mathcal{K}^b(\Lambda)}(T^\bullet)$. Since $p : E_T^{F, \Phi}(V, X) \rightarrow P$ is an epimorphism and $E_T^{F, \Phi}(V, X)$ is a projective Λ -module, there is a Λ -module homomorphism $u^0 : E_T^{F, \Phi}(V, X) \rightarrow E_T^{F, \Phi}(V, X)$ such that $u^0 p = pf^0$. Let $u^1 := f^1$ and $u^i = 0$ for all $i \neq 0, 1$. Then it follows from

$$u^0 E_T^{F, \Phi}(V, \bar{\alpha}) = u^0 pq = pf^0 q = pqf^1 = E_T^{F, \Phi}(V, \bar{\alpha})u^1$$

that $u^\bullet = (u^i)_{i \in \mathbb{Z}}$ is an endomorphism in $\text{End}_{\mathcal{K}^b(\Lambda)}(\tilde{T}^\bullet)$. By Lemma 2.2 (1), we can assume that $u^0 = \mu(x)$ and $u^1 = \mu(y)$ with $x = (x_i)_{i \in \Phi} \in E_T^{F, \Phi}(X)$ and $y = (y_i)_{i \in \Phi} \in E_T^{F, \Phi}(M_1 \oplus M)$. Now, it follows from $E_T^{F, \Phi}(V, \bar{\alpha})u^1 = u^0 E_T^{F, \Phi}(V, \bar{\alpha})$ that

$$(\bar{\alpha}y_i)_{i \in \Phi} = (x_i F^i \bar{\alpha})_{i \in \Phi}, \text{ that is, } \bar{\alpha}y_i = x_i F^i \bar{\alpha} \text{ for } i \in \Phi.$$

For each $i \in \Phi$, we can form the following commutative diagram in \mathcal{T} :

$$\begin{array}{ccccccc}
 X & \xrightarrow{\bar{\alpha}} & M_1 \oplus M & \xrightarrow{\bar{\beta}} & W & \xrightarrow{\bar{w}} & X[1] \\
 (*) \quad x_i \downarrow & & \downarrow y_i & & \downarrow h_i & & \downarrow x_i[1] \\
 F^i X & \xrightarrow{F^i \bar{\alpha}} & F^i(M_1 \oplus M) & \xrightarrow{F^i \bar{\beta}} & F^i W & \xrightarrow{(F^i \bar{w})\delta(F, i, X, 1)} & (F^i X)[1].
 \end{array}$$

for some morphism $h_i \in \text{Hom}_{\mathcal{T}}(W, F^i W)$. Thus, for each $f^\bullet \in \text{End}_{\mathcal{K}^b(\Lambda)}(T^\bullet)$, we get an element $h := (h_i)_{i \in \Phi} \in \Gamma$ which is $E_{\mathcal{T}}^{F, \Phi}(W)$ by definition. This leads us to defining the following correspondence:

$$\Theta : \text{End}_{\mathcal{K}^b(\Lambda)}(T^\bullet) \longrightarrow \bar{\Gamma} = \Gamma/J, \quad f^\bullet \mapsto h + J.$$

Claim 1. Θ is well-defined.

Proof. Suppose that $f^\bullet \in \text{End}_{\mathcal{K}^b(\Lambda)}(T^\bullet)$ is null-homotopic, that is, there is a map $r : E_{\mathcal{T}}^{F, \Phi}(V, M_1 \oplus M) \longrightarrow P$ such that $f^0 = qr$ and $f^1 = rq$. Since p is surjective and $E_{\mathcal{T}}^{F, \Phi}(V, M_1 \oplus M)$ is projective in $\Lambda\text{-mod}$, there is a map $s : E_{\mathcal{T}}^{F, \Phi}(V, M_1 \oplus M) \longrightarrow E_{\mathcal{T}}^{F, \Phi}(V, X)$ such that $sp = r$. Hence $(u^0 - pqs)p = u^0 p - pqs p = u^0 p - pqr = u^0 p - pf^0 = 0$ and $u^1 = rq = spq$. By the assumption $X \in \mathcal{Y}^{F, \Phi}(M)$, Lemma 2.2 (3) yields a map $t : M_1 \oplus M \longrightarrow X$ such that $s = E_{\mathcal{T}}^{F, \Phi}(V, t) = \mu(\underline{t})$. Therefore,

$$\mu(x - \underline{\alpha}t)p = (u^0 - E_{\mathcal{T}}^{\Phi}(V, \bar{\alpha})E_{\mathcal{T}}^{\Phi}(V, t))p = (u^0 - pqs)p = 0$$

and $\mu(y - t\underline{\alpha}) = u^1 - spq = 0$. Consequently, $\text{Im}(\mu(x - \underline{\alpha}t)) \subseteq I \cdot E_{\mathcal{T}}^{F, \Phi}(V, X)$ and $y - t\underline{\alpha} = 0$. Thus $y_i = 0$ for all $0 \neq i \in \Phi$ and $y_0 = t\underline{\alpha}$. By Lemma 3.4 (3), we have $x_i = 0$ for all $0 \neq i \in \Phi$ and $x_0 - \bar{\alpha}t = ab$ for some morphisms $a : X \longrightarrow M'$ and $b : M' \longrightarrow X$ with $M' \in \text{add}(M)$. Since $\bar{\alpha}$ is a left $\text{add}(M)$ -approximation of X , there is a morphism $c : M_1 \oplus M \longrightarrow M'$ such that $a = \bar{\alpha}c$. It follows that

$$x_0 = ab + \bar{\alpha}t = \bar{\alpha}cb + \bar{\alpha}t = \bar{\alpha}(cb + t).$$

Now we consider the commutative diagram $(*)$. Suppose $0 \neq i \in \Phi$. Then we have shown that $x_i = y_i = 0$. Hence $\bar{\beta}h_i = y_i F^i \bar{\beta} = 0$. This implies that h_i factorises through \bar{w} , and, consequently, that $h_i|_M = 0$ since $\bar{w}|_M = 0$. It follows from $h_i(F^i \bar{w})\delta(F, i, X, 1) = \bar{w}(x_i[1]) = 0$ that $h_i : W \rightarrow F^i W$ factorises through $F^i(M_1 \oplus M)$. Since $Y \in \mathcal{X}^{F, \Phi}(M)$, we get $h_i|_Y = 0$. Altogether, we have shown that $h_i = 0$ for all $0 \neq i \in \Phi$. Now consider the diagram $(*)$ in case $i = 0$. First, we have $\bar{\beta}h_0 = y_0 \bar{\beta} = t\underline{\alpha}\bar{\beta} = 0$, which means h_0 factorises through \bar{w} . Second, since $h_0 \bar{w} = \bar{w}(x_0[1]) = \bar{w}(\bar{\alpha}[1])(cb + t)[1] = 0$, the morphism h_0 factorises through $M_1 \oplus M$ which is in $\text{add}(M)$. Thus, $h \in J$ and $h + J$ is zero in $\bar{\Gamma}$. This shows that Θ is well-defined.

Claim 2. Θ is injective.

Proof. Suppose that $\Theta(f^\bullet) = h + J = 0 + J$. Then $h \in J$, that is, $h_i = 0$ for all $0 \neq i \in \Phi$, and h_0 factorises through both \bar{w} and $\text{add}(M)$. Suppose $h_0 = \bar{w}s$ for a morphism $s : X[1] \longrightarrow W$. For each $0 \neq i \in \Phi$, since $y_i F^i \bar{\beta} = \bar{\beta}h_i = 0$, the morphism $y_i : M_1 \oplus M \rightarrow F^i(M_1 \oplus M)$ factorises through $F^i X$, and consequently $y_i = 0$ for all $0 \neq i \in \Phi$ since $X \in \mathcal{Y}^{F, \Phi}(M)$. For each $0 \neq i \in \Phi$, it follows from $\bar{w}(x_i[1]) = h_i(F^i \bar{w})\delta(F, i, X, 1) = 0$ that $x_i[1]$ factorises through $(M_1 \oplus M)[1]$, or equivalently, the morphism $x_i : X \rightarrow F^i X$ factorises through $M_1 \oplus M$. Hence $x_i = 0$ for all $0 \neq i \in \Phi$ since $X \in \mathcal{Y}^{F, \Phi}(M)$. Now we consider the case $i = 0$. First, we have $y_0 \bar{\beta} = \bar{\beta}h_0 = \bar{\beta}\bar{w}s = 0$, which implies $y_0 = t\underline{\alpha}$ for a morphism $t : M_1 \oplus M \longrightarrow X$. Second, $(x_0 - \bar{\alpha}t)\bar{\alpha} = \bar{\alpha}y_0 - \bar{\alpha}t\bar{\alpha} = \bar{\alpha}y_0 - \bar{\alpha}y_0 = 0$. It follows that $(x_0 - \bar{\alpha}t)\alpha = 0$, and therefore $x_0 - \bar{\alpha}t$ factorises through $-w[-1]$. Since $h_0 : W \rightarrow W$ factorises through $\text{add}(M)$ and since $\bar{\beta} : M_1 \oplus M \rightarrow W$ is a right $\text{add}(M)$ -approximation of W , we see that h_0 factorises through $\bar{\beta}$, say $h_0 = r\bar{\beta}$ for some $r : W \rightarrow M_1 \oplus M$. Thus, $\bar{w}(x_0[1]) = h_0 \bar{w} = r\bar{\beta}\bar{w} = 0$, or equivalently, $(-\bar{w}[-1])x_0 = 0$. It follows that x_0 factorises through $M_1 \oplus M$. Since $\bar{\alpha}t$ also factorises through $M_1 \oplus M$, we see that

$x_0 - \bar{\alpha}t$ factorises through $\text{add}(M)$. Thus we have shown that $x_0 - \bar{\alpha}t$ factorises through both $\text{add}(M)$ and $-w[-1]$. Now, by Lemma 3.4 (3), we have $\text{Im}(\mu(x) - E_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha}t)) = \text{Im}(\mu(x - \bar{\alpha}t)) \subseteq I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$. Hence

$$p(f^0 - qE_{\mathcal{T}}^{F,\Phi}(V, t)p) = u^0p - pqE_{\mathcal{T}}^{F,\Phi}(V, t)p = (\mu(x) - E_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha}t))p = 0.$$

This implies that $f^0 = q(E_{\mathcal{T}}^{\Phi}(V, t)p)$ since p is surjective. Moreover, one can check that

$$f^1 = u^1 = \mu(y) = E_{\mathcal{T}}^{F,\Phi}(V, t)E_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha}) = (E_{\mathcal{T}}^{F,\Phi}(V, t)p)q.$$

Hence f^\bullet is null-homotopic, and consequently Θ is injective.

Claim 3. Θ is surjective.

Proof. Let $h = (h_i)_{i \in \Phi} \in \Gamma$ with $h_i : W \rightarrow F^iW$ for $i \in \Phi$. Since $\bar{\beta}$ is a right $(\text{add}(M), F, -\Phi)$ -approximation of W , we have a morphism $F^{-i}y_i : F^{-i}(M_1 \oplus M) \rightarrow M_1 \oplus M$ such that $(F^{-i}\bar{\beta})(F^{-i}h_i) = (F^{-i}y_i)\bar{\beta}$ for $i \in \Phi$. This means that there is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\bar{\alpha}} & M_1 \oplus M & \xrightarrow{\bar{\beta}} & W & \xrightarrow{\bar{w}} & X[1] \\ x_i \downarrow & & \downarrow y_i & & \downarrow h_i & & \downarrow x_i[1] \\ F^iX & \xrightarrow{F^i\bar{\alpha}} & F^i(M_1 \oplus M) & \xrightarrow{F^i\bar{\beta}} & F^iW & \xrightarrow{(F^i\bar{w})\delta(F, i, X, 1)} & F^iX[1]. \end{array}$$

Now, define $x := (x_i)_{i \in \Phi} \in E_{\mathcal{T}}^{F,\Phi}(X)$, $y := (y_i)_{i \in \Phi} \in E_{\mathcal{T}}^{F,\Phi}(M_1 \oplus M)$; $u^0 := \mu(x)$, $u^1 := \mu(y)$ and $u^j := 0$ for $j \neq 0, 1$. Then $u^\bullet := (u^i)_{i \in \mathbb{Z}}$ belongs to $\text{End}_{\mathcal{K}^b(\Lambda)}(\tilde{T}^\bullet)$. Since $u^0 : E_{\mathcal{T}}^{F,\Phi}(V, X) \rightarrow E_{\mathcal{T}}^{F,\Phi}(V, X)$ takes elements in $I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$ to elements in $I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$, the image of $I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$ under the map u^0p is zero, and consequently, there is a unique map $f^0 : P \rightarrow P$ such that $pf^0 = u^0p$. Now we have

$$p(f^0q - qu^1) = pf^0q - pqu^1 = u^0pq - pqu^1 = u^0E_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha}) - E_{\mathcal{T}}^{F,\Phi}(V, \bar{\alpha})u^1 = 0$$

Hence $f^0q = qu^1$ since p is surjective. Defining $f^1 = u^1$ and $f^j = 0$ for all $j \neq 0, 1$, we see that $f^\bullet = (f^i)_{i \in \mathbb{Z}}$ is an endomorphism in $\text{End}_{\mathcal{K}^b(\Lambda)}(T^\bullet)$ and $\Theta(f^\bullet) = h + J$. Thus Θ is surjective.

Claim 4. Θ is an R -algebra homomorphism.

Proof. The map Θ is R -linear, so it preserves addition. For multiplication, we take f^\bullet and g^\bullet in $\text{End}_{\mathcal{K}^b(\Lambda)}(T^\bullet)$. Let u^\bullet and v^\bullet be in $\text{End}_{\mathcal{K}^b(\Lambda)}(\tilde{T}^\bullet)$ such that $u^0p = pf^0$, $u^1 = f^1$, $v^0p = pg^0$ and $v^1 = g^1$. Suppose that $(u^0, u^1) = (\mu(x), \mu(y))$ and $(v^0, v^1) = (\mu(x'), \mu(y'))$ with $x, x' \in E_{\mathcal{T}}^{F,\Phi}(X)$ and $y, y' \in E_{\mathcal{T}}^{\Phi}(M_1 \oplus M)$. Let $h := (h_i)_{i \in \Phi}$ and $h' := (h'_i)_{i \in \Phi}$ be in Γ making the diagram (*) commutative, that is,

$$\begin{aligned} \bar{\beta}h_i &= y_i F^i \bar{\beta}, & \bar{w}(x_i[1]) &= h_i(F^i \bar{w})\delta(F, i, X, 1), \\ \bar{\beta}h'_i &= y'_i F^i \bar{\beta}, & \bar{w}(x'_i[1]) &= h'_i(F^i \bar{w})\delta(F, i, X, 1) \end{aligned}$$

for all $i \in \Phi$. Then, by definition, we have $\Theta(f^\bullet) = h + J$, $\Theta(g^\bullet) = h' + J$ and

$$\Theta(f^\bullet)\Theta(g^\bullet) = \left(\sum_{\substack{i, j \in \Phi \\ i+j=k}} h_i(F^i h'_j) \right)_{k \in \Phi} + J.$$

Now we calculate $\Theta(f^\bullet g^\bullet)$. Let $s^\bullet := u^\bullet v^\bullet$. Then $s^0p = pf^0g^0 = p(f^\bullet g^\bullet)^0$, $s^1 = f^1g^1 = (f^\bullet g^\bullet)^1$, and $(s^0, s^1) = (\mu(xx'), \mu(yy'))$, where $(xx')_k = \sum_{\substack{i, j \in \Phi \\ i+j=k}} x_i F^i x'_j$, and $(yy')_k = \sum_{\substack{i, j \in \Phi \\ i+j=k}} y_i F^i y'_j$. For each $k \in \Phi$, one has

to check that

$$(yy')_k F^k \bar{\beta} = \left(\sum_{\substack{i, j \in \Phi \\ i+j=k}} y_i F^i y'_j \right) F^k \bar{\beta} = \bar{\beta} \left(\sum_{\substack{i, j \in \Phi \\ i+j=k}} h_i F^i h'_j \right).$$

However, this follows from

$$\begin{aligned}
y_i(F^i y'_j)(F^{i+j} \bar{\beta}) &= y_i F^i (y'_j (F^j \bar{\beta})) \\
&= y_i F^i (\bar{\beta} h'_j) \\
&= y_i (F^i \bar{\beta}) (F^i h'_j) \\
&= \bar{\beta} h_i (F^i h'_j).
\end{aligned}$$

Similarly, for each $k \in \Phi$, we have

$$\left(\sum_{\substack{i,j \in \Phi \\ i+j=k}} h_i F^i h'_j \right) (F^k \bar{w}) \delta(F, i, X, 1) = \bar{w}((xx')_k[1]).$$

This means $\Theta(f^\bullet g^\bullet) = \left(\sum_{\substack{i,j \in \Phi \\ i+j=k}} h_i F^i h'_j \right)_{k \in \Phi} + J = \Theta(f^\bullet) \Theta(g^\bullet)$. Thus Θ is a ring homomorphism, and the proof of Theorem 3.1 is finished. \square

Before proceeding, we comment on the conditions in Theorem 3.1.

(a) Let $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ be a triangle in \mathcal{T} with $M_1 \in \text{add}(M)$, $X \in \mathcal{Y}^{F,\Phi}(M)$ and $Y \in \mathcal{X}^{F,\Phi}(M)$. If α is a left $(\text{add}(M), F, \Phi)$ -approximation of X , then $\text{Hom}_{\mathcal{T}}(X, F^i M) \simeq \text{Hom}_{\mathcal{T}}(M_1, F^i M)$ for $0 \neq i \in \Phi$. Similarly, if β is a right $(\text{add}(M), F, -\Phi)$ -approximation of Y , then $\text{Hom}_{\mathcal{T}}(M, F^i Y) = \text{Hom}_{\mathcal{T}}(M, F^i M_1)$ for $0 \neq i \in \Phi$. In particular, if M is an (F, Φ) -self-orthogonal object of \mathcal{T} , that is, $\text{Hom}_{\mathcal{T}}(M, F^i M) = 0$ for every $0 \neq i \in \Phi$, and if α is a left $(\text{add}(M), F, \Phi)$ -approximation of X and β is a right $(\text{add}(M), F, -\Phi)$ -approximation of Y , then $X \in \mathcal{X}^{F,\Phi}(M)$ and $Y \in \mathcal{Y}^{F,\Phi}(M)$.

(b) Under the conditions of Theorem 3.1, there are isomorphisms $\text{Hom}_{\mathcal{T}}(X, F^i X) \simeq \text{Hom}_{\mathcal{T}}(Y, F^i Y)$ for every $0 \neq i \in \Phi$. In fact, this follows from the following general statement:

Let \mathcal{T} be a triangulated category with a shift functor $[1]$. Suppose that F is a triangle functor from \mathcal{T} to itself, and that \mathcal{D} is a full subcategory of \mathcal{T} . Let i be a positive integer. Suppose that

$$X_j \xrightarrow{\alpha_j} D_j \xrightarrow{\beta_j} Y_j \longrightarrow X_j[1]$$

is a triangle in \mathcal{T} , such that α_j is a left $(\mathcal{D}, F, \{i\})$ -approximation of X_j , and that $\text{Hom}_{\mathcal{T}}(D', F^i(\beta_j)) : \text{Hom}_{\mathcal{T}}(D', F^i D_j) \rightarrow \text{Hom}_{\mathcal{T}}(D', F^i Y_j)$ is surjective for every $D' \in \mathcal{D}$ and $j = 1, 2$. If $\text{Hom}_{\mathcal{T}}(\mathcal{D}, F^i X_j) = 0 = \text{Hom}_{\mathcal{T}}(Y_j, F^i \mathcal{D})$ for $1 \leq j \leq 2$, then $\text{Hom}_{\mathcal{T}}(X_1, F^i X_2) \simeq \text{Hom}_{\mathcal{T}}(Y_1, F^i Y_2)$.

Proof. From the given two triangles the following exact commutative diagram can be formed:

$$\begin{array}{ccccccc}
& & & \text{Hom}_{\mathcal{T}}(D_1, F^i X_2) & \longrightarrow & \text{Hom}_{\mathcal{T}}(D_1, F^i D_2) & \\
& & & \downarrow & & \downarrow (\alpha_1, F^i D_2) & \\
& & & \text{Hom}_{\mathcal{T}}(X_1, F^i X_2) & \longrightarrow & \text{Hom}_{\mathcal{T}}(X_1, F^i D_2) & \\
& & & \downarrow & & \downarrow 0 & \\
\text{Hom}_{\mathcal{T}}(Y_1, F^i D_2) & \longrightarrow & \text{Hom}_{\mathcal{T}}(Y_1, F^i Y_2) & \longrightarrow & \text{Hom}_{\mathcal{T}}(Y_1, F^i X_2[1]) & \longrightarrow & \text{Hom}_{\mathcal{T}}(Y_1, F^i D_2[1]) \\
\downarrow & & \downarrow & & \downarrow & (*) & \downarrow \\
\text{Hom}_{\mathcal{T}}(D_1, F^i D_2) & \xrightarrow{(D_1, F^i(\beta_2))} & \text{Hom}_{\mathcal{T}}(D_1, F^i Y_2) & \xrightarrow{0} & \text{Hom}_{\mathcal{T}}(D_1, F^i X_2[1]) & \longrightarrow & \text{Hom}_{\mathcal{T}}(D_1, F^i D_2[1]).
\end{array}$$

Since $\text{Hom}_{\mathcal{T}}(Y_1, F^i D_2) = \text{Hom}_{\mathcal{T}}(D_1, F^i X_2) = 0$ by assumption and since $\text{Hom}_{\mathcal{T}}(\alpha_1, F^i D_2)$ and $\text{Hom}_{\mathcal{T}}(D_1, F^i \beta_2)$ are surjective by the property of approximation, the conclusion follows from the commutative square (*). \square

(c) Let $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ be an $\text{add}(M)$ -split triangle in \mathcal{T} . Define $V := X \oplus M$, $\Lambda_0 := \text{End}_{\mathcal{T}}(V)$, $W := M \oplus Y$, and $\Gamma_0 := \text{End}_{\mathcal{T}}(W)$. Let I and J be as defined in Theorem 3.1. Then the ideals I and J in Theorem 1.1 have the following characterisation:

(i) Let e be the idempotent in Γ_0 corresponding to the direct summand M of W . Then J is the submodule of the left Γ_0 -module $\Gamma_0 e \Gamma_0$, which is maximal with respect to $eJ = 0$.

(ii) Let f be the idempotent in Λ_0 corresponding to the direct summand M of V . Then I is the submodule of the right Λ_0 -module $\Lambda_0 f \Lambda_0$ which is maximal with respect to $If = 0$.

Proof. By Lemma 3.3, the sets I and J are ideals of Λ_0 and Γ_0 , respectively.

(i) Let $p_M : W \rightarrow M$ and $\lambda_M : M \rightarrow W$ be the canonical projection and injection, respectively. By definition, $e = p_M \lambda_M$. The set $\Gamma_0 e \Gamma_0$ is precisely the set of all endomorphisms of W that factorise through $\text{add}(M)$. The endomorphisms of W factorising through \bar{w} are those endomorphisms x that satisfy $\bar{\beta}x = 0$, and consequently $ex = p_M \lambda_M x = p_M(\bar{\beta}|_M)x = 0$. Hence J is a submodule of $\Gamma_0 \Gamma_0 e \Gamma_0$ with $eJ = 0$. Suppose that $\bar{J} \subseteq \Gamma_0 \Gamma_0 e \Gamma_0$ is another submodule containing J with $e\bar{J} = 0$. Then $e\bar{J} = 0$ implies $\text{Hom}_{\Gamma_0}(\text{Hom}_{\mathcal{T}}(W, M), \bar{J}) = 0$, and consequently $\text{Hom}_{\Gamma_0}(\text{Hom}_{\mathcal{T}}(W, M'), \bar{J}) = 0$ for all $M' \in \text{add}(M)$. For each $x \in \bar{J}$, the image of the morphism $\text{Hom}_{\mathcal{T}}(W, x)$ is contained in \bar{J} since \bar{J} is a left ideal of Γ_0 . Thus, the morphism $\text{Hom}_{\mathcal{T}}(W, \bar{\beta}x)$ is a Γ_0 -module morphism from $\text{Hom}_{\mathcal{T}}(W, M_1 \oplus M)$ to the image of $\text{Hom}_{\mathcal{T}}(W, x)$. Hence $\text{Hom}_{\mathcal{T}}(W, \bar{\beta}x) = 0$, and consequently $\bar{\beta}x = 0$. This implies $x \in J$. This proves (i).

(ii) The proof is similar to that of (i). \square

A special case of Theorem 3.1 is the following corollary.

Corollary 3.7. *Let Φ be an admissible subset of \mathbb{Z} , and let \mathcal{T} be a triangulated R -category with an auto-equivalence F , and let M be an object in \mathcal{T} . Suppose that $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ is an $\text{add}(M)$ -split triangle in \mathcal{T} , and suppose that X and Y both are in $\mathcal{X}^{F, \Phi}(M) \cap \mathcal{Y}^{F, \Phi}(M)$. Then $E_{\mathcal{T}}^{F, \Phi}(X \oplus M)/I$ and $E_{\mathcal{T}}^{F, \Phi}(M \oplus Y)/J$ are derived equivalent.*

The following special case of Theorem 3.1 is useful to construct explicit examples.

Corollary 3.8. *Let \mathcal{T} be a triangulated R -category with $[1]$ the shift functor, and let M be an object in \mathcal{T} . Suppose that $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ is a triangle in \mathcal{T} such that $M_1 \in \text{add}(M)$, and suppose that $X \in \mathcal{Y}^{n+1}(M)$ and $Y \in \mathcal{X}^{n+1}(M)$. Then, for any admissible subset Φ of \mathbb{N}_n , the algebras $E_{\mathcal{T}}^{\Phi}(X \oplus M)/I$ and $E_{\mathcal{T}}^{\Phi}(M \oplus Y)/J$ are derived equivalent.*

Proof. We show that β is a right $(\text{add}(M), -\Phi)$ -approximation of Y . Note that, for $i \in \Phi$, we always have $i + 1 \leq n + 1$. Hence $\text{Hom}_{\mathcal{T}}(M, X[i + 1]) = 0$ for $i \in \Phi$. Now apply $\text{Hom}_{\mathcal{T}}(M[-i], -)$ with $i \in \Phi$ to the triangle $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$:

$$\cdots \rightarrow \text{Hom}_{\mathcal{T}}(M[-i], M_1) \rightarrow \text{Hom}_{\mathcal{T}}(M[-i], Y) \rightarrow \text{Hom}_{\mathcal{T}}(M[-i], X[1]) \rightarrow \cdots$$

Because of $\text{Hom}_{\mathcal{T}}(M[-i], X[1]) = \text{Hom}_{\mathcal{T}}(M, X[i + 1]) = 0$, the map β is a right $(\text{add}(M), -\Phi)$ -approximation of Y .

Similarly, it follows from $\text{Ext}_{\mathcal{T}}^{i+1}(Y, M) = 0$ for $i \in \Phi$ that α is a left $(\text{add}(M), \Phi)$ -approximation of X . Now Corollary 3.8 follows from Theorem 3.1. \square

Another special case of Theorem 3.1 is that $I = 0$ and $J = 0$. Here is a condition when the ideals I and J in Theorem 3.1 vanish.

Proposition 3.9. *Let $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ be an $\text{add}(M)$ -split triangle in \mathcal{T} . Define $V := X \oplus M$, $\Lambda_0 := \text{End}_{\mathcal{T}}(V)$, $W := M \oplus Y$, and $\Gamma_0 := \text{End}_{\mathcal{T}}(W)$. Let I' be the ideal of Λ_0 consisting of all $f : V \rightarrow V$ that factorises through $\tilde{w}[-1] : Y[-1] \rightarrow V$, and let J' be the ideal of Γ_0 consisting of all $g : W \rightarrow W$ that factorises through $\tilde{w} : W \rightarrow X[1]$.*

(1) Suppose that Λ_0 is an Artin algebra. If $\text{add}(\text{top}_{\Lambda_0} \text{Hom}_{\mathcal{T}}(V, X)) \cap \text{add}(\text{top}_{\Lambda_0} D\Lambda_0) = 0$, then $I' = 0$.

(2) Suppose that Γ_0 is an Artin algebra. If $\text{add}(\text{top}_{\Gamma_0} \text{Hom}_{\mathcal{T}}(W, Y)) \cap \text{add}(\text{soc}(\Gamma_0 \Gamma_0)) = 0$, then $J' = 0$.

Proof. We prove (1). The proof of (2) is similar to that of (1), and we omit it.

We have a triangle $Y[-1] \xrightarrow{-\tilde{w}[-1]} V \xrightarrow{\tilde{\alpha}} M_1 \oplus M \xrightarrow{\tilde{\beta}} Y$, apply $\text{Hom}_{\mathcal{T}}(-, V)$ to this triangle, and get the following exact sequence of right Λ_0 -modules:

$$\text{Hom}_{\mathcal{T}}(M_1 \oplus M, V) \longrightarrow \text{Hom}_{\mathcal{T}}(V, V) \longrightarrow C \longrightarrow 0,$$

where C is the cokernel of $\text{Hom}_{\mathcal{T}}(\tilde{\alpha}, V)$. Now, applying $\text{Hom}_{\Lambda_0^{\text{op}}}(\text{Hom}_{\mathcal{T}}(M, V), -)$ to the above exact sequence, we get another exact sequence which is isomorphic to the following exact sequence:

$$\text{Hom}_{\mathcal{T}}(M_1 \oplus M, M) \xrightarrow{(\tilde{\alpha}, M)} \text{Hom}_{\mathcal{T}}(V, M) \longrightarrow \text{Hom}_{\Lambda_0^{\text{op}}}(\text{Hom}_{\mathcal{T}}(M, V), C) \longrightarrow 0.$$

Since $\tilde{\alpha}$ is a left $\text{add}(M)$ -approximation of V , the map $\text{Hom}_{\mathcal{T}}(\tilde{\alpha}, M)$ is surjective, and consequently $\text{Hom}_{\Lambda_0^{\text{op}}}(\text{Hom}_{\mathcal{T}}(M, V), C) = 0$. So, the right Λ_0 -module C has no composition factors in $\text{top}(\text{Hom}_{\mathcal{T}}(M, V))$, and that C has composition factors only in $\text{top}(\text{Hom}_{\mathcal{T}}(X, V))$. This is equivalent to saying that the Λ_0 -module $D(C)$ has composition factors only in $\text{soc}(D\text{Hom}_{\mathcal{T}}(X, V))$ which is isomorphic to $\text{top}(\text{Hom}_{\mathcal{T}}(V, X))$.

Let $x : V \rightarrow V$ be an element in $I' \subseteq \Lambda_0$. Then x factorises through $-\tilde{w}[-1]$, or equivalently, $x\tilde{\alpha} = 0$. This implies that $(D\text{Hom}_{\mathcal{T}}(x, V))(D\text{Hom}_{\mathcal{T}}(\tilde{\alpha}, V)) = 0$. Thus the image of $D\text{Hom}_{\mathcal{T}}(x, V)$ is contained in the kernel of $D\text{Hom}_{\mathcal{T}}(\tilde{\alpha}, V)$, which is isomorphic to $D(C)$. Therefore, if $D\text{Hom}_{\mathcal{T}}(x, V) \neq 0$, then the top of the image of $D\text{Hom}_{\mathcal{T}}(x, V)$ is contained in $\text{add}(\text{top}_{\Lambda_0} \text{Hom}_{\mathcal{T}}(V, X)) \cap \text{add}(\text{top}_{\Lambda_0} D\Lambda_0) = 0$, this is a contradiction. Thus we must have $\text{Hom}_{\mathcal{T}}(x, V) = 0$. Since $\text{Hom}_{\mathcal{T}}(-, V)$ is a duality from $\text{add}(V)$ to $\Lambda_0^{\text{op}}\text{-proj}$, we obtain $x = 0$. Thus $I' = 0$. \square

Remark. (1) if we substitute “ $\text{add}(M)$ -split” for “left $(\text{add}(M), \Phi)$ -approximation” and “right $(\text{add}(M), -\Phi)$ -approximation” in Proposition 3.9, and if we consider $E_{\mathcal{T}}^{\Phi}(V)$ and $E_{\mathcal{T}}^{\Phi}(W)$ instead of Λ_0 and Γ_0 , then Proposition 3.9 is still true. The proof is almost the same.

(2) By definition, there are inclusions $I \subseteq I'$ and $J \subseteq J'$. Sometimes it is easy to verify that I' and J' vanish if the algebras Λ_0 and Γ_0 are described by quivers with relations.

For the derived category of an abelian category, the following result provides an explicit example for $I = 0 = J$.

Proposition 3.10. *Let \mathcal{A} be an abelian category, and let M be an object of \mathcal{A} . Suppose that $0 \rightarrow X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow 0$ is an exact sequence in \mathcal{A} with $M_1 \in \text{add}(M)$. Consider the induced triangle $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ in $\mathcal{D}^b(\mathcal{A})$. Then the ideals I and J defined in Theorem 3.1 vanish.*

Proof. Every exact sequence $0 \rightarrow X \rightarrow M_1 \rightarrow Y \rightarrow 0$ in \mathcal{A} gives rise to a triangle $X \rightarrow M_1 \rightarrow Y \rightarrow X[1]$ in $\mathcal{D}^b(\mathcal{A})$. Now we show that the exactness of the given sequence in \mathcal{A} implies that the two ideals I and J in Theorem 3.1 are equal to zero. Since I is contained in $\text{End}_{\mathcal{D}^b(\mathcal{A})}(X \oplus M)$, it is sufficient to show that if a morphism $x : X \oplus M \rightarrow X \oplus M$ factorises through $\text{add}(M)$ and $\tilde{w}[-1]$, then $x = 0$. Let x be such a morphism. Then we see immediately that $x\tilde{\alpha} = 0$ in $\mathcal{D}^b(\mathcal{A})$. Since \mathcal{A} is fully embedded in $\mathcal{D}^b(\mathcal{A})$, we also have $x\tilde{\alpha} = 0$ in \mathcal{A} . Consequently, $x = 0$ since $\tilde{\alpha}$ is injective in \mathcal{A} . Thus $I = 0$. Dually, we can show $J = 0$. Hence Proposition 3.10 holds true. \square

As an immediate application of the proof of Theorem 3.1 together with a result on derived equivalences in [18], we have the following corollary.

Corollary 3.11. *We keep all assumptions of Theorem 3.1. If $\overline{\Lambda}$ and $\overline{\Gamma}$ both are left coherent rings (for example, if Φ is finite and $\mathcal{T} = \mathcal{D}^b(A)$ with A a finite dimensional algebra over a field), then $\text{fin.dim}(\overline{\Lambda}) - 1 \leq \text{fin.dim}(\overline{\Gamma}) \leq \text{fin.dim}(\overline{\Lambda}) - 1$, where $\text{fin.dim}(\overline{\Lambda})$ stands for the finitistic dimension of $\overline{\Lambda}$.*

Recall that, given a ring S with identity, the *finitistic dimension* of S is defined to be the supremum of the projective dimensions of finitely generated S -modules of finite projective dimension.

Since the map q in the proof of Theorem 3.1 is not always injective, the tilting complex T^\bullet is not, in general, isomorphic in $\mathcal{D}^b(E_{\mathcal{T}}^{F,\Phi}(V)/I)$ to a tilting module. Thus the derived equivalence presented in Theorem 3.1 is not given by a tilting module in general (in contrast with the situation of Theorem 2.3). In fact, it is easy to see that the derived equivalence in Theorem 3.1 is given by a tilting module if the kernel of $E_{\mathcal{T}}^{F,\Phi}(V, \alpha)$ is $I \cdot E_{\mathcal{T}}^{F,\Phi}(V, X)$.

Moreover, a small additive category may be embedded into an abelian category of coherent functors (see [16, Chapter IV, Section 2]). This will, however, not in general turn a \mathcal{D} -split sequence in the additive category into an exact sequence in the abelian category since otherwise the sequence would split, and therefore cannot provide a triangle in the derived category of the abelian category. Consequently, Theorem 2.3 cannot be obtained from Theorem 3.1 by taking $\Phi = \{0\}$ and embedding an additive category into an abelian category.

Finally, we mention that Theorem 3.1 generalises the result [10, Proposition 5.1] by choosing $\Phi = \{0\}$. Indeed, under the conditions of [10, Proposition 5.1], the ideals I and J in Theorem 3.1 vanish. Theorem 3.1 covers various other situations, some of which will be discussed in the next section.

4 Φ -Yoneda algebras in some explicit situations

In this section, we shall describe some natural habitats for Theorem 3.1 and relate it to several widely used concepts that fit with or simplify the assumptions of Theorem 3.1. Throughout, we choose F to be the shift functor of the triangulated category considered.

We note that Alex Dugas, in independent work [5] that also is motivated by [10], has constructed derived equivalent pairs of symmetric algebras. As explained in [5] (Remark (3) in section 4) his examples appear in our framework, too.

4.1 Derived categories of Artin algebras

A first consequence of Theorem 3.1 is the following result for $\mathcal{T} = \mathcal{D}^b(A)$ with A an Artin R -algebra.

Theorem 4.1. *Let Φ be an admissible subset of \mathbb{N} , let M be an A -module, and let $0 \rightarrow X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow 0$ be an exact sequence in $A\text{-mod}$ with α a left $(\text{add}(M), \Phi)$ -approximation of X and β a right $(\text{add}(M), -\Phi)$ -approximation of Y in $\mathcal{D}^b(A)$ such that $X \in \mathcal{D}^\Phi(M)$ and $Y \in \mathcal{X}^\Phi(M)$. Then the perforated Yoneda algebras $E_A^\Phi(X \oplus M)$ and $E_A^\Phi(M \oplus Y)$ are derived equivalent.*

Proof. This is a consequence of Theorem 3.1 and Proposition 3.10 if we take $\mathcal{T} = \mathcal{D}^b(A)$. \square

Under the assumptions of Theorem 4.1, the higher cohomology groups $\text{Ext}_A^i(X, X)$ of X is isomorphic to the higher cohomology groups $\text{Ext}_A^i(Y, Y)$ of Y for each $0 \neq i \in \Phi$. This follows from the comment (b) before Corollary 3.7.

When requiring additional orthogonality conditions on X and Y in Theorem 4.1, we get the following corollary.

Corollary 4.2. *Let Φ be an admissible subset of \mathbb{N} , let M be an A -module, and let $0 \rightarrow X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow 0$ be an $\text{add}(M)$ -split sequence in $A\text{-mod}$ such that $X, Y \in \mathcal{X}^\Phi(M) \cap \mathcal{Y}^\Phi(M)$. Then the perforated Yoneda algebras $E_A^\Phi(X \oplus M)$ and $E_A^\Phi(M \oplus Y)$ are derived equivalent.*

Proof. This follows immediately from Corollary 3.7 and Proposition 3.10. \square

If the orthogonality conditions in Corollary 4.2 hold for \mathbb{N}_n or \mathbb{N} , then we get the following consequence.

Corollary 4.3. *Suppose that M is an A -module. Let $0 \rightarrow X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow 0$ be an $\text{add}(M)$ -split sequence in $A\text{-mod}$ such that $X, Y \in \mathcal{X}^n(M) \cap \mathcal{Y}^n(M)$ for n a positive number or infinity. Then, for any admissible subset Φ of \mathbb{N}_n , the perforated Yoneda algebras $E_A^\Phi(X \oplus M)$ and $E_A^\Phi(M \oplus Y)$ are derived equivalent.*

The following result shows that the orthogonality conditions are related to the concepts of *short cycle* and *short chain* in $A\text{-mod}$ (see [1, Chapter IX, p.313]). Recall that a short cycle of length 2 from an indecomposable module X to X is a sequence of non-zero radical homomorphisms $X \xrightarrow{f} M \xrightarrow{g} X$ with M indecomposable; and a short chain is a sequence of non-zero radical homomorphisms $X \xrightarrow{f} M \xrightarrow{g} D\text{Tr}(X)$ with X indecomposable.

Corollary 4.4. *Let A be an Artin algebra, and let $0 \rightarrow X \rightarrow M \rightarrow Y \rightarrow 0$ be an Auslander-Reiten sequence in $A\text{-mod}$. Suppose neither X nor Y lies on a short cycle of length 2, nor on a short chain. Then the trivial extension of $\text{End}_A(X \oplus M)$ by the bimodule $\text{Ext}_A^1(X, X) \oplus \text{Ext}_A^1(M, M)$ is derived equivalent to the trivial extension of $\text{End}_A(M \oplus Y)$ by the bimodule $\text{Ext}_A^1(Y, Y) \oplus \text{Ext}_A^1(M, M)$.*

Proof. An Auslander-Reiten sequence $0 \rightarrow X \rightarrow M \rightarrow Y \rightarrow 0$ is always an $\text{add}(M)$ -split sequence. Since Y does not lie on a short cycle, the Auslander-Reiten formula $D\overline{\text{Hom}}_A(\text{Tr } D(X), M) \simeq \text{Ext}_A^1(M, X) \simeq D\overline{\text{Hom}}_A(X, D\text{Tr}(M))$ (see [1, p.131]) implies $\text{Ext}_A^1(M, X) = 0$. Moreover, X not lying on a short cycle implies $\text{Ext}_A^1(Y, M) = 0$. Similarly, the Auslander-Reiten formula yields that $\text{Ext}_A^1(X, M) = 0$ - since X does not lie on a short chain - and that $\text{Ext}_A^1(M, Y) = 0$ - since Y does not lie on a short chain. Thus Corollary 4.4 follows from Corollary 4.3 when $n = 1$. \square

The next corollary is a consequence of Corollary 4.3.

Corollary 4.5. *Let A be an Artin algebra, and let X be an A -module such that $\text{Ext}_A^i(X, A) = 0$ for all $1 \leq i < n + 2$ with n a fixed positive integer or infinity. Then, for any admissible subset Φ of \mathbb{N}_n , the perforated Yoneda algebras $E_A^\Phi(A \oplus X)$ and $E_A^\Phi(A \oplus \Omega(X))$ are derived equivalent.*

Proof. If $\text{Ext}_A^i(X, A) = 0$ for a fixed $i \geq 1$, then $0 \rightarrow \Omega^i(X) \rightarrow P_{i-1} \rightarrow \Omega^{i-1}(X) \rightarrow 0$ is an $\text{add}(A)$ -split sequence in $A\text{-mod}$, where P_i is a projective cover of $\Omega^i(X)$. Using this fact, Corollary 4.5 follows immediately from Corollary 4.3. \square

The condition $\text{Ext}_A^i(X, A) = 0$ on X in Corollary 4.5 is related to the context of the *Generalised Nakayama Conjecture*. This states that if an A -module T satisfies $\text{Ext}_A^i(A \oplus T, A \oplus T) = 0$ for all $i > 0$ then T should be projective. The above Corollary 4.5 (or [10, Theorem 1.1]) describes the shape of the syzygy modules $\Omega^i(X)$: If X is indecomposable and non-projective and satisfies $\text{Ext}_A^i(X, A) = 0$ for all $i > 0$, then, for each $j \geq 0$, there is an indecomposable non-projective module L_j such that $\Omega^j(X) \simeq L_j^{m_j}$ for an integer $m_j > 0$.

In Corollary 4.5, there are isomorphisms $\text{Ext}_A^i(X, X) \simeq \text{Ext}_A^i(\Omega(X), \Omega(X))$ for all $i \geq 1$. Thus the algebras $E_A^\Phi(A \oplus X)$ and $E_A^\Phi(A \oplus \Omega(X))$ are the extensions of $\text{End}_A(A \oplus X)$ and $\text{End}_A(A \oplus \Omega(X))$ by the same ideal $E_A^{\Phi \setminus \{0\}}(X, X)$, respectively. The algebras $E_A^\Phi(X \oplus M)$ and $E_A^\Phi(M \oplus Y)$ in Corollary 4.3,

however, are the extensions of $\text{End}_A(X \oplus M)$ and $\text{End}_A(M \oplus Y)$ by possibly different ideals $E_A^{\Phi \setminus \{0\}}(M) \oplus E_A^{\Phi \setminus \{0\}}(X)$ and $E_A^{\Phi \setminus \{0\}}(M) \oplus E_A^{\Phi \setminus \{0\}}(Y)$, respectively.

Recall that a module $M \in A\text{-mod}$ is called *reflexive* if the evaluation map

$$\alpha_M : M \rightarrow M^{**} := \text{Hom}_{A^{\text{op}}}(\text{Hom}_A(M, A), A_A)$$

is an isomorphism of modules.

Corollary 4.6. *Let M be a reflexive A -module. Then, for any subset $0 \in \Phi \subseteq \{0, 1\}$, the perforated Yoneda algebras $E_A^\Phi(D(A_A) \oplus D\text{Tr}(M))$ and $E_A^\Phi(D(A_A) \oplus \Omega^{-1}(D\text{Tr}(M)))$ are derived equivalent, where Ω^{-1} is the co-syzygy operator.*

Proof. By [1, IV, Proposition 3.2], the kernel and cokernel of the evaluation map α_M are $\text{Ext}_{A^{\text{op}}}^1(\text{Tr}(M), A)$ and $\text{Ext}_{A^{\text{op}}}^2(\text{Tr}(M), A)$, respectively. As $E_A^\Phi(U) \simeq E_A^\Phi(D(U))^{\text{op}}$ for any A -module U , Corollary 4.6 follows from Corollary 4.5 for right modules. \square

A special case of Corollary 4.5, is a result on self-injective algebras that has been obtained in [11, Corollary 3.14]):

Corollary 4.7. *If A is a self-injective Artin algebra, then, for any admissible subset Φ of \mathbb{N} , the perforated Yoneda algebras $E_A^\Phi(A \oplus X)$ and $E_A^\Phi(A \oplus \Omega(X))$ are derived equivalent.*

Another concept related to the Generalised Nakayama Conjectures and to modules being projective and injective, is the *dominant dimension* of an algebra or a module.

Suppose that A is an Artin R -algebra. By definition, the *dominant dimension* of A is greater than or equal to n if in the minimal injective resolution of ${}_A A$:

$$0 \longrightarrow A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow I_n \longrightarrow \cdots,$$

the first n injective A -modules I_0, \dots, I_{n-1} are projective. In this case we write $\text{dom.dim}(A) \geq n$. Let C_i be the i -th cosyzygy of A , that is, the cokernel of the map $I_{i-1} \rightarrow I_i$.

For an A -module X , we define $a(X)$ to be the number of non-isomorphic indecomposable direct summands of M . The *self-injective measure* of A is defined to be the number $m(A) := a(A) - a(I_0)$, where I_0 is an injective hull of A . Thus, if A is self-injective, then $m(A) = 0$. If $\text{dom.dim}(A) \geq 1$, then A is self-injective if and only if $m(A) = 0$. So the Nakayama conjecture can be reformulated as: If $\text{dom.dim}(A) = \infty$, then $m(A) = 0$.

Corollary 4.8. *Let A be an Artin algebra, and let T be the direct sum of all non-isomorphic indecomposable projective-injective A -modules.*

- (1) *If $\text{dom.dim}(A) \geq n \geq 2$, then $\text{End}_A(T \oplus C_i)$ is derived equivalent to A for $1 \leq i < n$.*
- (2) *If $\text{dom.dim}(A) \geq n + 1 < \infty$, then $m(A) = a(C_n)$.*

Proof. Since the sequence $0 \rightarrow C_{i-1} \rightarrow I_i \rightarrow C_i \rightarrow 0$ is an $\text{add}(I_i)$ -split sequence (or an $\text{add}(T)$ -split sequence), the orthogonality conditions in Corollary 4.3 are trivially satisfied. Derived equivalence preserves the number of non-isomorphic simple modules. Therefore, Corollary 4.8 follows now from Corollary 4.3. Here we also use the observation that $\text{add}(C_i) \cap \text{add}(I_j) = \{0\}$ for all $0 \leq i, j \leq n$. Alternatively, one can also use Lemma 2.3 to prove this corollary. \square

Examples of algebras of dominant dimension at least n can be obtained in the following way: Let A be a self-injective algebra and X an A -module. If $\text{Ext}_A^i(X, X) = 0$ for all $1 \leq i \leq n$, then $\text{dom.dim}(\text{End}_A(A \oplus X)) \geq n + 2$.

Finally, we turn to Auslander-regular algebras.

Let Λ be a k -algebra over a field k . Recall that Λ is called *Auslander-regular* if Λ has finite global dimension and satisfies the Gorenstein condition: if $p < q$ are non-negative integers and M is a finitely generated (left or right) Λ -module, then $\text{Ext}_{\Lambda}^p(N, \Lambda) = 0$ for every submodule N of $\text{Ext}_{\Lambda}^q(M, \Lambda)$. Here, if M is a right Λ -module, then N is a left Λ -module. Let $j(M)$ be the minimal number $r \geq 0$ such that $\text{Ext}_{\Lambda}^r(M, \Lambda) \neq 0$. Then for any submodule N of $\text{Ext}_{\Lambda}^{j(M)}(M, \Lambda)$, we have $\text{Ext}_{\Lambda}^i(N, \Lambda) = 0$ for $0 < i < j(M)$. Thus:

Corollary 4.9. *Let Λ be an Auslander-regular k -algebra, and M a finitely generated right Λ -module. Then, for any submodule X of $\text{Ext}_{\Lambda}^{j(M)}(M, \Lambda)$, and any admissible subset Φ of $\mathbb{N}_{j(M)-2}$, the algebras $E_{\Lambda}^{\Phi}(\Lambda \oplus X)$ and $E_{\Lambda}^{\Phi}(\Lambda \oplus \Omega(X))$ are derived equivalent.*

4.2 Frobenius categories

Let \mathcal{A} be a *Frobenius* abelian category, that is, \mathcal{A} is an abelian category with enough projective objects and enough injective objects such that the projective objects coincides with the injective objects. We denote by $\underline{\mathcal{A}}$ the stable category of \mathcal{A} modulo projective objects. It is shown in [8] that $\underline{\mathcal{A}}$ is a triangulated category, in which the shift functor $[1]$ is just the co-syzygy functor Ω^{-1} , and the triangles in $\underline{\mathcal{A}}$ are all induced by short exact sequences in \mathcal{A} . For each morphism $f : U \rightarrow V$ in \mathcal{A} , we denote by \underline{f} the image of f under the canonical functor from \mathcal{A} to $\underline{\mathcal{A}}$. Note that the objects of $\underline{\mathcal{A}}$ are the same as those of \mathcal{A} .

Lemma 4.10. *Let Φ be an admissible subset of \mathbb{N} , and let M, X , and Y be objects in \mathcal{A} . Then*

(1) *For arbitrary $0 \neq i \in \mathbb{N}$ and $U, U' \in \mathcal{A}$, there is an isomorphism*

$$\text{Hom}_{\mathcal{D}^b(\mathcal{A})}(U, U'[i]) \simeq \text{Hom}_{\underline{\mathcal{A}}}(U, U'[i]),$$

which is functorial in U and U' ;

(2) *A monomorphism $\alpha : X \rightarrow M_1$ in \mathcal{A} is a left $(\text{add}(M), \Phi)$ -approximation of X in $\mathcal{D}^b(\mathcal{A})$ if and only if $\underline{\alpha}$ is a left $(\text{add}(M), \Phi)$ -approximation of X in $\underline{\mathcal{A}}$;*

(3) *An epimorphism $\beta : M_2 \rightarrow Y$ in \mathcal{A} is a right $(\text{add}(M), -\Phi)$ -approximation of Y in $\mathcal{D}^b(\mathcal{A})$ if and only if $\underline{\beta}$ is a right $(\text{add}(M), -\Phi)$ -approximation of Y in $\underline{\mathcal{A}}$.*

Proof. (1) For $0 \neq i \in \mathbb{N}$, the isomorphisms

$$\text{Hom}_{\mathcal{D}^b(\mathcal{A})}(U, U'[i]) \simeq \text{Ext}_{\mathcal{A}}^i(U, U') \simeq \text{Hom}_{\underline{\mathcal{A}}}(U, \Omega^{-i}U') = \text{Hom}_{\underline{\mathcal{A}}}(U, U'[i]).$$

are functorial in U and U' . Thus (1) follows.

(2) First, let $0 \neq i$ be in Φ . By (1), there is a commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}^b(\mathcal{A})}(M_1, M[i]) & \xrightarrow{(\alpha, M[i])} & \text{Hom}_{\mathcal{D}^b(\mathcal{A})}(X, M[i]) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Hom}_{\underline{\mathcal{A}}}(M_1, M[i]) & \xrightarrow{(\underline{\alpha}, M[i])} & \text{Hom}_{\underline{\mathcal{A}}}(X, M[i]). \end{array}$$

Thus, the map $\text{Hom}_{\underline{\mathcal{A}}}(\underline{\alpha}, M[i])$ is surjective if and only if $\text{Hom}_{\mathcal{D}^b(\mathcal{A})}(\alpha, M[i])$ is surjective. Now we consider the case $i = 0$. If every morphism from X to M in \mathcal{A} factorises through α , then every morphism from X to M in $\underline{\mathcal{A}}$ factorises through $\underline{\alpha}$. Conversely, assume that every morphism from X to M in $\underline{\mathcal{A}}$ factorises through $\underline{\alpha}$. Let $f : X \rightarrow M$ be a morphism in \mathcal{A} . Then $\underline{f} = \underline{\alpha}h$ for some $h : M_1 \rightarrow M$ in \mathcal{A} . Thus $f - \alpha h$ in \mathcal{A} factorises through a projective object P , say $f - \alpha h = st$ for some $s : X \rightarrow P$ and $t : P \rightarrow M$ in \mathcal{A} . Since P is also injective and α is a monomorphism, there is some morphism $r : M_1 \rightarrow P$ such that $s = \alpha r$. Altogether, $f = \alpha h + st = \alpha h + \alpha rt = \alpha(h + rt)$ factorises through α . Thus the statement (2) follows. The proof of (3) is similar to that of (2). \square

Proposition 4.11. *Let Φ be an admissible subset of \mathbb{N} . Suppose that \mathcal{A} is a Frobenius abelian category, that M is an object in \mathcal{A} , and that $0 \rightarrow X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow 0$ is a short exact sequence in \mathcal{A} with $M_1 \in \text{add}(M)$ such that the induced triangle $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow X[1]$ in $\underline{\mathcal{A}}$ satisfies the conditions in Theorem 3.1. Then the algebras $E_{\mathcal{D}(\mathcal{A})}^{\Phi}(M \oplus Y)$ and $E_{\mathcal{D}(\mathcal{A})}^{\Phi}(X \oplus M)$ are derived equivalent.*

Proof. This follows from Lemma 4.10 and Proposition 3.10. \square

Corollary 4.12. *Suppose that \mathcal{A} is a Frobenius abelian category and M is an object in \mathcal{A} . Let $0 \rightarrow X \rightarrow M_1 \rightarrow Y \rightarrow 0$ with $M_1 \in \text{add}(M)$ be a short exact sequence in \mathcal{A} such that the induced triangle in $\underline{\mathcal{A}}$ is an $\text{add}(M)$ -split triangle. Then $\text{End}_{\mathcal{A}}(M \oplus Y)$ and $\text{End}_{\mathcal{A}}(X \oplus M)$ are derived equivalent.*

Proof. Taking $\Phi := \{0\}$, the corollary follows from Proposition 4.11. \square

Remark. If \mathcal{A} is a Frobenius (not necessarily abelian) category, then Corollary 4.12 is still true. For the precise definition of a Frobenius category, we refer the reader to [8].

The module category of a self-injective Artin algebra is a Frobenius abelian category. In this case, we have the following corollary.

Corollary 4.13. *Let A be a self-injective algebra, and let M be an A -module. Suppose $X \rightarrow M_1 \rightarrow Y \rightarrow X[1]$ is an $\text{add}(M)$ -split triangle in $A\text{-mod}$. Then $\text{End}_A(A \oplus M \oplus X)$ and $\text{End}_A(A \oplus M \oplus Y)$ are derived equivalent.*

Proof. Since all triangles in $A\text{-mod}$ are induced by short exact sequences in $A\text{-mod}$, there is a short exact sequence $0 \rightarrow X \rightarrow M_1 \oplus P \rightarrow Y \rightarrow 0$ in $A\text{-mod}$ with P projective such that the induced triangle is isomorphic to the given triangle $X \rightarrow M_1 \rightarrow Y \rightarrow X[1]$ in $A\text{-mod}$. The triangle $X \rightarrow M_1 \oplus P \rightarrow Y \rightarrow X[1]$ also is an $\text{add}(A \oplus M)$ -split triangle in $A\text{-mod}$. The corollary then follows from Corollary 4.12. \square

4.3 Calabi-Yau categories

The theory of Calabi-Yau and cluster categories provides very natural contexts for our construction of derived equivalences.

Let k be a field, and let \mathcal{T} be a k -linear triangulated category which is Hom-finite, that is, the Hom-space $\text{Hom}_{\mathcal{T}}(X, Y)$ is finite dimensional over k for all X and Y in \mathcal{T} .

Recall that \mathcal{T} is called $(n+1)$ -Calabi-Yau for some non-negative integer n if there is a natural isomorphism between $D\text{Hom}_{\mathcal{T}}(X, Y)$ and $\text{Hom}_{\mathcal{T}}(Y, X[n+1])$ for all X and Y in \mathcal{T} , where $D = \text{Hom}_k(-, k)$ is the usual duality. It follows that $\mathcal{X}_{\mathcal{T}}^n(M) = \mathcal{Y}_{\mathcal{T}}^n(M)$ for $M \in \mathcal{T}$. (See [13] for more information on Calabi-Yau categories.)

Note that if $\Phi = \{0, 1, \dots, n\}$, then $n - i \in \Phi$ for each $i \in \Phi$.

Lemma 4.14. *Let $\Phi = \{0, 1, \dots, n\}$. Suppose that \mathcal{T} is an $(n+1)$ -Calabi-Yau triangulated category, and that M is an object in \mathcal{T} . Let $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow X[1]$ be a triangle in \mathcal{T} with $M_1 \in \text{add}(M)$. Then:*

- (1) *The morphism α is a left $(\text{add}(M), \Phi)$ -approximation of X if and only if the morphism β is a right $(\text{add}(M), -\Phi)$ -approximation of Y ;*
- (2) *If α is a left $(\text{add}(M), \Phi)$ -approximation of X and if M is n -self-orthogonal, then $X \in \mathcal{X}^n(M) \cap \mathcal{Y}^n(M)$ and $Y \in \mathcal{X}^n(M) \cap \mathcal{Y}^n(M)$.*

Proof. We will abbreviate $\text{Hom}_{\mathcal{T}}(-, -)$ by $(-, -)$. First we assume that α is a left $(\text{add}(M), \Phi)$ -approximation of X . Now, for each $i \in \Phi$, there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
(M[-i], M_1) & \xrightarrow{(M[-i], \beta)} & (M[-i], Y) & & & & \\
\downarrow \simeq & & \downarrow \simeq & & & & \\
(M, M_1[i]) & \xrightarrow{(M, \beta[i])} & (M, Y[i]) & \longrightarrow & (M, X[i+1]) & \xrightarrow{(M, \alpha[i+1])} & (M, M_1[i+1]) \\
& & & & \downarrow \simeq & & \downarrow \simeq \\
& & & & D(X, M[n-i]) & \xrightarrow{D(\alpha, M[n-i])} & D(M_1, M[n-i]).
\end{array}$$

Since $n-i$ is in Φ , and since α is a left $(\text{add}(M), \Phi)$ -approximation of X , the map $(\alpha, M[n-i])$ is surjective, and consequently $D(\alpha, M[n-i])$ is injective. Hence $(M, \alpha[i+1])$ is injective, and therefore $(M[-i], \beta)$ is surjective. This shows that β is a right $(\text{add}(M), -\Phi)$ -approximation of Y . The proof of the other implication in (1) can be done similarly.

(2) It follows from (1) and the comment before Corollary 3.7 that $X \in \mathcal{X}_{\mathcal{T}}^{\Phi}(M)$ and $Y \in \mathcal{Y}_{\mathcal{T}}^{\Phi}(M)$. Since \mathcal{T} is $(n+1)$ -Calabi-Yau, we have $(M, X[i]) \simeq D(X, M[n+1-i]) = 0$, and $(M, Y[i]) \simeq D(Y, M[n+1-i]) = 0$ for all $0 \neq i \in \Phi$. Thus $X \in \mathcal{Y}_{\mathcal{T}}^{\Phi}(M)$ and $Y \in \mathcal{X}_{\mathcal{T}}^{\Phi}(M)$. \square

Corollary 4.15. *Let $\Phi = \{0, 1, \dots, n\}$, and let \mathcal{T} be an $(n+1)$ -Calabi-Yau triangulated category. Suppose that M is n -self-orthogonal and $Y \in \mathcal{Y}^n(M)$. Let $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ be a triangle in \mathcal{T} with β a right $\text{add}(M)$ -approximation of Y . Then the algebras $E_{\mathcal{T}}^{\Phi}(M \oplus X)/I$ and $E_{\mathcal{T}}^{\Phi}(M \oplus Y)/J$ are derived equivalent, where I and J are defined as in Theorem 3.1.*

Proof. Since $Y \in \mathcal{Y}_{\mathcal{T}}^{\Phi}(M)$, for each $0 \neq i \in \Phi$, the map $(M[-i], M_1) \rightarrow (M[-i], Y) = 0$ induced by β is surjective. Taking into account that β is a right $\text{add}(M)$ -approximation of Y , we see that β is, in fact, a right $(\text{add}(M), -\Phi)$ -approximation of Y . By Proposition 4.14 (1), the map α is a left $(\text{add}(M), \Phi)$ -approximation of X . Since M is n -self-orthogonal, the proof can be finished by applying Proposition 4.14 (2) and Corollary 3.7 to the triangle. \square

Corollary 4.15 is related to mutations in a Calabi-Yau category. Here are some definitions from [12].

Let \mathcal{T} be an $(n+1)$ -Calabi-Yau category. An object T in \mathcal{T} is called an n -cluster tilting object if T is n -self-orthogonal, and if any $X \in \mathcal{T}$ with $\text{Ext}_{\mathcal{T}}^i(T, X) = 0$ for $1 \leq i \leq n$ is in $\text{add}(T)$. The object T is called *basic* if the multiplicity of each indecomposable direct summand of T is one.

Let T be an n -cluster basic tilting object in an $(n+1)$ -Calabi-Yau category \mathcal{T} , and Y a direct summand of T , that is, $T = Y \oplus M$. Let $\beta : M_1 \rightarrow Y$ be a minimal right $\text{add}(M)$ -approximation of Y , and let

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \longrightarrow X[1]$$

be a triangle containing β . Note that we allow Y to be decomposable, and that X is indecomposable if and only if Y is indecomposable. The object $X \oplus M$ is called the *left mutation* of T at Y . In the case of tilting modules, X is called a *tilting complement* to M in the literature (see, for example, [9]). It was pointed out in [12] that the left mutation of T at Y is again an n -cluster tilting object (for some special cases, see [3, 7], and also [15, p.314]). In fact, this can be seen in the following way: The proof of Corollary 4.15 and comment (b) on the conditions of Theorem 3.1 imply that $T' := M \oplus X$ is n -self-orthogonal. Moreover, let $X' \in \mathcal{X}^n(T')$ and consider a triangle $X' \xrightarrow{\alpha'} M' \rightarrow Y' \rightarrow X'[1]$ with α' a left $\text{add}(M)$ -approximation of X' . Then $Y' \in \mathcal{X}^n(T)$ by Lemma 4.14 and the comment (b). Thus $Y' \in \text{add}(T)$, $X' \in \text{add}(T')$, and $T' := X \oplus M$ is again an n -cluster tilting object in \mathcal{T} . The notion of a right mutation of T at Y is dual.

Usually, $\text{End}_{\mathcal{T}}(X \oplus M)$ and $\text{End}_{\mathcal{T}}(M \oplus Y)$ are not derived equivalent. When they are derived equivalent may be an interesting question. Here is a sufficient condition.

Corollary 4.16. *Let $\Lambda := \text{End}_T(X \oplus M)$ and $\Gamma := \text{End}_T(M \oplus Y)$. Then*

(1) *$\text{End}_T(X \oplus M)/I$ and $\text{End}_T(M \oplus Y)/J$ are derived equivalent.*

(2) *Suppose that Y is indecomposable. Let S_X be the simple Λ -module corresponding to X , and let S_Y be the simple Γ -module corresponding to Y . Suppose that S_Y is not a submodule of Γ , and S_X is not a quotient of $D(\Lambda)$. Then Λ and Γ are derived equivalent.*

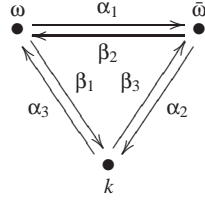
Proof. Statement (1) is a direct consequence of Corollary 4.15, and (2) follows from (1) and Proposition 3.9. \square

Remark. Consider a 2-Calabi-Yau category, and assume that $\text{Ext}_T^1(S_Y, S_Y) = 0$. Then we re-obtain the result [14, Theorem 5.3] from Corollary 4.16 (2).

5 Examples

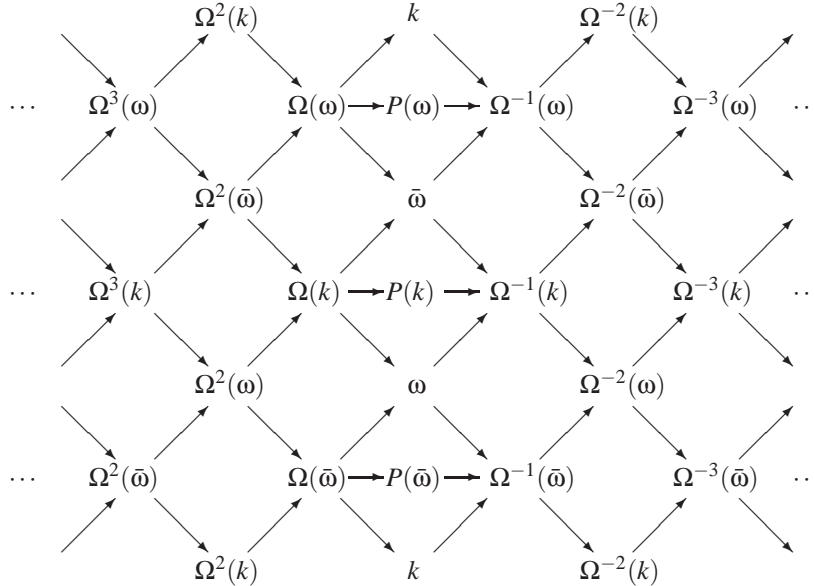
First, we present an explicit example which satisfies all conditions in Theorem 3.1.

Example 1. Let k be an algebraically closed field of characteristic 2, and let $A := kA_4$ be the group algebra of the alternating group A_4 . Then there are three simple A -modules, which are denoted k , ω , and $\bar{\omega}$, respectively. Their projective covers are $P(k)$, $P(\omega)$ and $P(\bar{\omega})$, respectively. It is well known that kA_4 is Morita equivalent to the following algebra given by quiver



and relations $\alpha_i \beta_{i+1} - \beta_i \alpha_{i+2} = \alpha_i \alpha_{i+1} = \beta_i \beta_{i-1} = 0$, where the subscripts are considered modulo 3.

As this algebra is symmetric, the Auslander-Reiten translation $D\text{Tr}$ is just the second syzygy Ω^2 . The Auslander-Reiten quiver of this algebra is well-known to have a component of the following form:



Consider the Auslander-Reiten sequence

$$0 \longrightarrow \Omega^3(\omega) \longrightarrow \Omega^2(k) \oplus \Omega^2(\bar{\omega}) \longrightarrow \Omega(\omega) \longrightarrow 0.$$

Let $X = \Omega^3(\omega)$, $Y = \Omega(\omega)$, and $M = \Omega^2(k) \oplus \Omega^2(\bar{\omega})$. This sequence provides an Auslander-Reiten triangle in the triangulated category $A\text{-mod}$:

$$X \longrightarrow M \longrightarrow Y \longrightarrow X[1].$$

We shall check that this triangle satisfies the conditions of Theorem 3.1.

We choose $\Phi = \{0, 1\}$ and $F = [1]$. Since this is an Auslander-Reiten triangle in $A\text{-mod}$, the map $X \rightarrow M$ is a left $(\text{add}(M), \Phi)$ -approximation of X , and the map $M \rightarrow Y$ is a right $(\text{add}(M), -\Phi)$ -approximation of Y (see the example at the end of Section 2). It follows from the above Auslander-Reiten quiver of A that $\text{Ext}_A^1(M, X) \simeq \underline{\text{Hom}}_A(M, \Omega^{-1}(X)) \simeq \underline{\text{Hom}}_A(\Omega^2(k) \oplus \Omega^2(\bar{\omega}), \Omega^2(\omega)) = 0$ and $\text{Ext}_A^1(Y, M) \simeq \underline{\text{Hom}}_A(Y, \Omega^{-1}(M)) = \underline{\text{Hom}}_A(\Omega(\omega), \Omega(k) \oplus \Omega(\bar{\omega})) = 0$. Thus the above triangle in $A\text{-mod}$ satisfies all conditions in Theorem 3.1, and therefore, by Proposition 4.11, the algebras $E_A^\Phi(M \oplus X)$ and $E_A^\Phi(M \oplus Y)$ are derived equivalent.

Furthermore, we have $\text{Ext}_A^1(M, M) \simeq \underline{\text{Hom}}_A(M, \Omega^{-1}M) \simeq \underline{\text{Hom}}_A(\Omega(k) \oplus \Omega(\bar{\omega}), k \oplus \bar{\omega})$. There is an epimorphism from $\Omega(k)$ to $\bar{\omega}$ and an epimorphism from $\Omega(\bar{\omega})$ to k . The latter cannot factorise through a projective module, we get $\dim_k \text{Ext}_A^1(M, M) = 2$. Moreover, there is an epimorphism from $\Omega(k)$ to ω and an epimorphism from $\Omega(\bar{\omega})$ to ω . This implies $\dim_k \text{Ext}_A^1(M, Y) = 2$. Similarly, $\dim_k \text{Ext}_A^1(X, M) = 2$. Note that all the indecomposable modules appearing in the Auslander-Reiten triangle are 1-self-orthogonal. A more precise calculation shows that $\dim_k E_A^\Phi(M \oplus X) = 33$ and $\dim_k E_A^\Phi(M \oplus Y) = 21$.

The following example shows that the Ext-orthogonality conditions in Corollary 4.2 and therefore in Theorem 3.1 cannot be dropped.

Example 2. Let A be the algebra (over a field k) given by the following quiver with relations:

$$\begin{array}{c} \bullet \xleftarrow{\beta} \bullet \xrightarrow{\alpha} \bullet \\ 1 \quad 2 \end{array} \quad \alpha^2 = 0 = \alpha\beta.$$

This example is in a class of examples constructed by Small [22]. The algebra A is of finite representation type, its finitistic dimension equals one, while the finitistic dimension of the opposite algebra A^{op} is zero.

We denote by $S(i)$ and $P(i)$ the simple and projective modules corresponding to the vertex i , respectively. Let M_i be the quotient module of $P(2)$ by $S(i)$, and $M := M_1 \oplus M_2 = D(A_A)$, where D is the usual duality. Then there is an Auslander-Reiten sequence

$$0 \longrightarrow X := P(2) \longrightarrow M \longrightarrow S(2) =: Y \longrightarrow 0.$$

This is an $\text{add}(M)$ -split sequence in $A\text{-mod}$.

If we take $\Phi = \{0, 1\}$, then $E_A^\Phi(X \oplus M) = \text{End}_A(X \oplus M)$. An easy calculation shows that $\text{End}_A(X \oplus M)$ is a quasi-hereditary algebra, and thus has finite global dimension. The algebra $E_A^\Phi(M \oplus Y)$ contains a loop which is given by the short exact sequence induced by the loop α at the vertex 2. Thus it has infinite global dimension by the 'no loops theorem'. It follows that $E_A^\Phi(X \oplus M)$ and $E_A^\Phi(M \oplus Y)$ cannot be derived equivalent since derived equivalences preserve the finiteness of global dimensions. Also, one can see that $\text{Ext}_A^i(X, M) = 0 = \text{Ext}_A^1(M, X)$ and $\text{Ext}_A^i(Y, M) = 0 \neq \text{Ext}_A^1(M, Y)$ for $i \geq 1$. This example shows that the orthogonality conditions in Corollary 4.2 cannot be omitted. Moreover, it shows that the result in [10, Theorem 1.1] cannot be extended from endomorphism algebras to Φ -Yoneda algebras without any additional conditions.

A A two functors version of Theorem 1.1

In Theorem 3.1, there is only one functor F involved. When working with the derived category of a hereditary algebra, or the stable category of a self-injective algebra, or the derived category of coherent sheaves of a projective variety over \mathbb{C} , apart from the shift functor there are other prominent functors, for example the Auslander-Reiten translation $D\text{Tr}$. To have available a general statement of construction of derived equivalences, which is similar to Theorem 3.1, we define Φ -perforated Yoneda algebras for two functors over a triangulated category, and formulate a two-functor version of Theorem 3.1. In this appendix, we summarise the ingredients for a possible generalisation of Theorem 3.1. The proof of this generalisation is analogous to that of Theorem 3.1, but more technical and tedious. So we omit it here.

Let Φ be a subset of $\mathbb{N} \times \mathbb{N}$, which we consider as a semigroup with ordinary addition. Let \mathcal{T} be a triangulated R -category with shift functor [1], and let X be an object in \mathcal{T} .

Suppose that F and G are two triangle functors from \mathcal{T} to itself, such that FG is naturally isomorphic to GF . For X in \mathcal{T} , let $\delta(i, j, X) : F^j G^i X \rightarrow G^i F^j X$ be an isomorphism induced from the natural transformation $FG \sim GF$. Then we define

$$E_{\mathcal{T}}^{F,G,\Phi}(X) := \bigoplus_{(i,j) \in \Phi} \text{Hom}_{\mathcal{T}}(X, G^i F^j X),$$

with elements of the form $(f_{i,j})_{(i,j) \in \Phi}$, where $f_{i,j} : X \rightarrow G^i F^j X$. The multiplication on $E_{\mathcal{T}}^{F,G,\Phi}(X)$ is given by

$$(f_{i,j})_{(i,j) \in \Phi} \cdot (g_{i,j})_{(i,j) \in \Phi} = \left(\sum_{\substack{(p,q),(u,v) \in \Phi \\ (u+p, v+q) = (i,m) \in \Phi}} f_{u,v}(G^u F^v g_{p,q})(G^u \delta(p, v, F^q X)) \right)_{(i,m) \in \Phi \times \Psi}.$$

If F and G are invertible, then Φ can be chosen a subset of $\mathbb{Z} \times \mathbb{Z}$.

A general model for the above definition is: Given a bi-graded algebra $\Lambda = \bigoplus_{i,j \in \mathbb{Z}} \Lambda_{i,j}$, we define $\Lambda(\Phi) = \bigoplus_{(i,j) \in \Phi} \Lambda_{i,j}$, and a multiplication by $a_{i,j} \cdot a_{p,q} = a_{i,j} a_{p,q}$ if $(i+p, j+q) \in \Phi$, and zero otherwise.

If Φ is admissible, for example, Φ is the cartesian product of two admissible sets in \mathbb{Z} , then $\Lambda(\Phi)$ is an associative algebra. So, we have to check that $E_{\mathcal{T}}^{F,G}(X) := \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(X, G^i F^j X)$ is an associative

algebra with respect to the above multiplication. This can be based on the following lemma.

Lemma A.1. *Suppose F and G are two triangle functors from \mathcal{T} to itself such that FG is naturally isomorphic to GF . For any triangle functor L from \mathcal{T} to itself, there is a natural isomorphism $\delta(i, j, L) : F^j G^i L \rightarrow G^i F^j L$ for all $i, j \geq 0$ such that, for $p, q, r, s \in \mathbb{N}$,*

- (1) $\delta(p+q, r, L) = \delta(p, r, G^q L)(G^p \delta(q, r, L));$
- (2) $\delta(p, r+s, L) = (F^s \delta(p, r, L))\delta(p, s, F^r L).$

Proof. For functors L_1 and L_2 from \mathcal{T} to itself, we define $L_1 \delta(1, 1, L_2) : L_1 F G L_2 \rightarrow L_1 G F L_2$ to be the induced natural isomorphism from the functor $L_1 F G L_2$ to the functor $L_1 G F L_2$. So, $\delta(1, 1, 1_{\mathcal{T}})$ is just the given natural isomorphism from FG to GF . Now we shall construct inductively a natural isomorphism $\delta(i, j, L)$ from $F^j G^i L$ to $G^i F^j L$ for all non-negative integers i and j and functors L from \mathcal{T} to itself.

If $i = 0$ or $j = 0$, then $F^j G^i L = G^i F^j L$, and we define $\delta(i, j, L)$ to be the identity natural transformation. For each positive integer $j > 1$, we assume that $\delta(1, j-1, L)$ is defined. Now we define

$$\delta(1, j, L) := (F \delta(1, j-1, L)) \delta(1, 1, F^{j-1} L).$$

For each positive integer $i > 1$, assume that $\delta(i-1, j, L)$ is defined. We define

$$\delta(i, j, L) := \delta(1, j, G^{i-1}L)(G\delta(i-1, j, L)).$$

(1) It is straightforward to check that (1) holds for $p+q \leq 2$. We shall prove (1) by induction on $p+q$. Now assume that $p+q > 2$. Then we have

$$\begin{aligned} \delta(p+q, r, L) &= \delta(1, r, G^{p+q-1}L)(G\delta(p+q-1, r, L)) \quad (\text{by definition}) \\ &= \delta(1, r, G^{p+q-1}L)G\left(\delta(p-1, r, G^qL)(G^{p-1}\delta(q, r, L))\right) \quad (\text{by induction}) \\ &= \left(\delta(1, r, G^{p+q-1}L)(G\delta(p-1, r, G^qL))\right)(G^p\delta(q, r, L)) \\ &= \delta(p, r, G^qL)(G^p\delta(q, r, L)) \quad (\text{by definition}). \end{aligned}$$

This proves (1).

(2) We first prove (2) for $p = 0, 1$. If $p = 0$, then (2) is clearly true. Now suppose $p = 1$. We shall show (2) by induction on $r+s$. In fact, if $r+s \leq 2$, it is straightforward to check (2). Now we assume that $r+s > 2$. Then we have

$$\begin{aligned} \delta(1, r+s, L) &= (F\delta(1, r+s-1, L))\delta(1, 1, F^{r+s-1}L) \quad (\text{by definition}) \\ &= F\left((F^{s-1}\delta(1, r, L))\delta(1, s-1, F^rL)\right)\delta(1, 1, F^{r+s-1}L) \quad (\text{by induction}) \\ &= (F^s\delta(1, r, L))\left((F\delta(1, s-1, F^rL))\delta(1, 1, F^{r+s-1}L)\right) \\ &= (F^s\delta(1, r, L))\delta(1, s, F^rL) \quad (\text{by definition}). \end{aligned}$$

This proves (2) for $p = 1$. Now assume $p > 1$. Then

$$\begin{aligned} \delta(p, r+s, L) &= \delta(1, r+s, G^{p-1}L)(G\delta(p-1, r+s, L)) \quad (\text{by definition}) \\ &= (F^s\delta(1, r, G^{p-1}L))\delta(1, s, F^rG^{p-1}L)G\left((F^s\delta(p-1, r, L))\delta(p-1, s, F^rL)\right) \quad (\text{by induction}) \\ &= (F^s\delta(1, r, G^{p-1}L))\left(\delta(1, s, F^rG^{p-1}L)(GF^s\delta(p-1, r, L))\right)(G\delta(p-1, s, F^rL)). \end{aligned}$$

Since $\delta(1, s, F^rG^{p-1}L)$ is a natural transformation from $F^sGF^rG^{p-1}L$ to $GF^sF^rG^{p-1}L$, the following diagram of natural transformations is commutative:

$$\begin{array}{ccc} F^sGF^rG^{p-1}L & \xrightarrow{\delta(1, s, F^rG^{p-1}L)} & GF^sF^rG^{p-1}L \\ F^sG\delta(p-1, r, L) \downarrow & & \downarrow GF^s\delta(p-1, r, L) \\ F^sGG^{p-1}F^rL & \xrightarrow{\delta(1, s, G^{p-1}F^rL)} & GF^sG^{p-1}F^rL. \end{array}$$

Hence

$$\begin{aligned} \delta(p, r+s, L) &= (F^s\delta(1, r, G^{p-1}L))\left(\delta(1, s, F^rG^{p-1}L)(GF^s\delta(p-1, r, L))\right)(G\delta(p-1, s, F^rL)) \\ &= (F^s\delta(1, r, G^{p-1}L))\left((F^sG\delta(p-1, r, L))\delta(1, s, G^{p-1}F^rL)\right)(G\delta(p-1, s, F^rL)) \\ &= F^s\left(\delta(1, r, G^{p-1}L)(G\delta(p-1, r, L))\right)\left(\delta(1, s, G^{p-1}F^rL)(G\delta(p-1, s, F^rL))\right) \\ &= (F^s\delta(p, r, L))\delta(p, s, F^rL). \end{aligned}$$

This proves (2). \square

Remark. If, in addition, F and G are invertible, then Lemma A.1 remains valid for i, j, p, q, r and s any integers.

Let \mathcal{D} be a full subcategory of \mathcal{T} , and X an object of \mathcal{T} . A morphism $f : X \rightarrow D$ with $D \in \mathcal{D}$ is called a left $(\mathcal{D}, F, G, \Phi)$ -approximation of X if $\text{Hom}_{\mathcal{T}}(f, G^iF^jD') : \text{Hom}_{\mathcal{T}}(D, G^iF^jD') \longrightarrow \text{Hom}_{\mathcal{T}}(X, G^iF^jD')$

is surjective for every object $D' \in \mathcal{D}$ and $(i, j) \in \Phi$. Dually, we define the right $(\mathcal{D}, F, G, \Phi)$ -approximation of X .

Given a triangle $0 \rightarrow X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ in \mathcal{T} with $M_1 \in \text{add}(M)$ for a fixed $M \in \mathcal{T}$, we define $\tilde{w}[-1] = (-w[-1], 0) : Y[-1] \rightarrow X \oplus M$, $\tilde{w} = (0, w)^T$, where $(0, w)^T$ stands for the transpose of the matrix $(0, w)$, and

$$\begin{aligned} I &:= \{x = (x_{i,j}) \in E_{\mathcal{T}}^{F,G,\Phi}(X \oplus M) \mid x_{i,j} = 0 \text{ for } (0,0) \neq (i,j) \in \Phi, \text{ and} \\ &\quad x_{0,0} \text{ factors through } \text{add}(M) \text{ and } \tilde{w}[-1]\}, \\ J &:= \{y = (y_{i,j}) \in E_{\mathcal{T}}^{F,G,\Phi}(M \oplus Y) \mid y_{i,j} = 0 \text{ for } (0,0) \neq (i,j) \in \Phi, \text{ and} \\ &\quad y_{0,0} \text{ factors through } \text{add}(M) \text{ and } \tilde{w}\}. \end{aligned}$$

Now, Theorem 1.1 generalises as follows:

Theorem A.2. *Let Φ be an admissible subset of $\mathbb{Z} \times \mathbb{Z}$, and let \mathcal{T} be a triangulated R -category, and let M be an object in \mathcal{T} . Assume that there are two invertible triangle functors F and G from \mathcal{T} to itself such that FG is naturally isomorphic to GF by $\delta : FG \rightarrow GF$. Suppose that $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ is a triangle in \mathcal{T} such that α is a left $(\text{add}(M), F, G, \Phi)$ -approximation of X and β is a right $(\text{add}(M), F, G, -(\Phi))$ -approximation of Y . If $\text{Hom}_{\mathcal{T}}(M, G^i F^j X) = 0 = \text{Hom}_{\mathcal{T}}(Y, G^i F^j(M))$ for $(0,0) \neq (i,j) \in \Phi$, then $E_{\mathcal{T}}^{F,G,\Phi}(X \oplus M)/I$ and $E_{\mathcal{T}}^{F,G,\Phi}(M \oplus Y)/J$ are derived equivalent.*

Taking $G = id$, we recover Theorem 3.1. Taking $G = [1]$ and $F = id$ yields a result on “Ext-algebras”. A.2.

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